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Least deviation of logarithmic derivatives of algebraic polynomials from zero

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Abstract

We study least deviation of logarithmic derivatives of real-valued algebraic polynomials with a fixed root from zero on the segment $[-1; 1]$ in the uniform norm with various weights $w(x)$. A final solution is given for $w(x) = \sqrt{1-x^2}$, and an asymptotically precise one is found for $w(x) \equiv 1$. As a corollary, new inequalities of Markov–Bernstein type are obtained for a special class of polynomials.

Keywords: Logarithmic derivatives of algebraic polynomials; Best approximation; Alternance; Markov–Bernstein type inequalities

1. Introduction

Logarithmic derivatives of algebraic polynomials (abbreviated to *l.d.*s in what follows) are rational functions of the form

$$\rho_0(z) \equiv 0, \qquad \rho_n(z) = \frac{P_n'(z)}{P_n(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}, \quad z, z_k \in \mathbb{C},\ n \in \mathbb{N}, \tag{1}$$

where $P_n(z) = \prod_{k=1}^{n}(z - z_k)$ and n is the *degree* of the *l.d.* Note that *l.d.*s are also widely called *simple partial fractions*.

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In rational approximation theory, attention to *l.d.*s was paid by works of Macintyre and Fuchs [24], Gonchar [15] and Dolzhenko [14]. Later on, approximating properties of *l.d.*s were studied by Korevaar [22] and Chui [5], who proposed a construction of *l.d.*s for approximation of analytic functions belonging to the Bergman–Bers class on simply connected domains. Besides, the poles of the *l.d.*s were chosen on the boundaries of the domains. Such an approximation was motivated by the fact that *l.d.*s specified plane electrostatic fields, consequently, it could be considered as a determination of locations z_k of electrons creating a given field to a high accuracy.

Further investigations of approximating properties of *l.d.*s were initiated by Gorin [16], who brought up a problem of least deviation of *l.d.*s with a fixed pole from zero on the real line. At various times this was studied by many mathematicians (see [7] and the references therein). Then the following analogue of classical Mergelyan's theorem about polynomial approximations was proved for *l.d.*s with free poles: every function continuous on the compact set $K \subset \mathbb{C}$ with connected complement and analytic in the interior of K can be approximated uniformly on K with *l.d.*s [11]. In many respects this result entailed current intensive study of approximation properties of *l.d.*s and several their modifications on bounded and unbounded subsets of the complex plane (see [2,3,6,8–10,13,17–21,23,27,28]).

Hereinafter we consider only real-valued *l.d.*s $\rho_n = \rho_n(x), x \in \mathbb{R}$ (sets of their poles are symmetric with respect to the real line). Moreover, we set

$$\|\rho_n\| := \max_{x\in I} |\rho_n(x)|, \qquad \|\rho_n\|^* := \max_{x\in I} |\sqrt{1-x^2}\rho_n(x)|, \quad I := [-1; 1].$$

Main results of our paper are devoted to the following problem stated by Danchenko [9].

Among all l.d.s of degree less than or equal to n with a fixed pole find the l.d.s $\widetilde{\rho}_n$ *and* $\widetilde{\rho}_n^*$ *of least deviation from zero on I in the norms* $\|\cdot\|$ *and* $\|\cdot\|^*$.

This extremal problem is one of possible analogues of classical Chebyshev's question about monic polynomials of least deviation[1] from zero on I and an analogue of Gorin's problem stated above in the case of a finite segment. We solve it in the class, being denoted by $\mathfrak{R}_n(a)$, of (real-valued) *l.d.*s with the fixed real pole $x = a$. Note that all results for I can be extended to the case of any segment with the substitution $y = \mu x + \nu,\ \mu > 0$. At that it should be taken into account that $\rho_n(x) = \mu\rho_n(y)$, therefore $\|\rho_n\|_{I_{\mu,\nu}} = \mu^{-1}\|\rho_n\|_I$, where $I_{\mu,\nu} := [-\mu+\nu; \mu+\nu]$.

2. *L.d.*s of least deviation from zero on *I* in the norm $\|\cdot\|^*$

In [9] and then in [12] the following *l.d.* was constructed:

$$\varsigma_n(A; x) = \frac{2nw}{w^2-1}\frac{A(w^{2n}-1)}{(Aw^n-1)(w^n-A)}, \qquad x = \frac{1}{2}\left(w + \frac{1}{w}\right), \quad w = e^{i\varphi}, \tag{2}$$

where $A > 1, x \in I, \varphi \in \mathbb{R}$. It was also shown there that the function $\sqrt{1-x^2}\varsigma_n(A; x)$ has an alternance consisting of n points in I, $\|\varsigma\|^* = 2nA(A^2-1)^{-1}$, and that all its poles belong to the ellipse

$$E_p := \left\{z : z = p\ \cos t + \sqrt{p^2-1}\ \sin t\ i,\ t \in [0; 2\pi),\ p > 1\right\}, \tag{3}$$

[1] Let us recall several definitions. Let $f = f(x)$ be a continuous function in I. *Least deviation* of *l.d.*s ρ_n from f on I is the quantity $\inf_{\rho_n} \|f - \rho_n\|$, where the infimum is taken over all *l.d.*s $\rho_n = \rho_n(x)$. The *l.d.* $\widetilde{\rho}_n$, least deviating from f on I, is called a *l.d. of best uniform approximation of f on I*. It is said that m pairwise distinct points t_k in I form *a (Chebyshev) alternance* of the difference $f - \rho_n$ on I if $f(t_k) - \rho_n(t_k) = \pm(-1)^k\|f - \rho_n\|,\ k = \overline{1, m}$.