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Estimates for n-widths of sets of smooth functions on the torus $\mathbb{T}^d$

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Abstract

In this paper, we investigate n-widths of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda, \Lambda_* : L^p(\mathbb{T}^d) \to L^q(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$ for a function λ defined on the interval $[0, \infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$. In the first part, upper and lower bounds are established for n-widths of general multiplier operators. In the second part, we apply these results to the specific multiplier operators $\Lambda^{(1)} = \left\{|\mathbf{k}|^{-\gamma}(\ln|\mathbf{k}|)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(1)}_* = \left\{|\mathbf{k}|_*^{-\gamma}(\ln|\mathbf{k}|_*)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(2)} = \left\{e^{-\gamma|\mathbf{k}|^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda^{(2)}_* = \left\{e^{-\gamma|\mathbf{k}|_*^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ for $\gamma, r > 0$ and $\xi \ge 0$. We have that $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are sets of finitely differentiable functions on $\mathbb{T}^d$, in particular, $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are Sobolev-type classes if $\xi = 0$, and $\Lambda^{(2)}U_p$ and $\Lambda^{(2)}_*U_p$ are sets of infinitely differentiable ($0 < r < 1$) or analytic ($r = 1$) or entire ($r > 1$) functions on $\mathbb{T}^d$, where U_p denotes the closed unit ball of $L^p(\mathbb{T}^d)$. In particular, we prove that, the estimates for the Kolmogorov n-widths $d_n(\Lambda^{(1)}U_p, L^q(\mathbb{T}^d))$, $d_n(\Lambda^{(1)}_*U_p, L^q(\mathbb{T}^d))$, $d_n(\Lambda^{(2)}U_p, L^q(\mathbb{T}^d))$ and $d_n(\Lambda^{(2)}_*U_p, L^q(\mathbb{T}^d))$ are order sharp in various important situations.

Keywords: Torus; Width; Multiplier; Smooth function

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1. Introduction

In [2,3,12,15,13,14,16,17] techniques were developed to obtain estimates for n-widths of multiplier operators defined on two-points homogeneous spaces: $\mathbb{S}^d$, $\mathbb{P}^d(\mathbb{R})$, $\mathbb{P}^d(\mathbb{C})$, $\mathbb{P}^d(\mathbb{H})$, P^{16} (Cay). In this paper, we obtain estimates for n-widths of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ defined on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$, for a function λ defined on $[0,\infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$.

The first result concerning asymptotic estimates for Sobolev classes on the circle is contained in the paper of Kolmogorov [11], where it is proved that $d_n(B_2^{(r)}, L^2) \asymp n^{-r}$. Later, in 1952, Rudin proves in [26] which seems to have been the first asymptotic result for $d_n(B_p^{(r)}, L^q)$, with $p < q$ and it was generalized by Stechkin two years later in [28]. After that, estimates for Sobolev classes $B_p^{(r)}$ were proved for several other cases by several mathematicians, among which we highlight Gluskin, Ismagilov, Maiorov, Makovoz and Scholz in [7,8,19,20,27]. In 1977, Kashin was able to complete the picture for $d_n(B_p^{(r)}, L^q)$, with $1 \le p, q \le \infty$ in [9,10]. Several techniques were applied in these different cases, among which we can emphasize a discretization technique due to Maiorov and the Borsuk theorem.

For estimates of n-widths of sets of analytic or entire functions, other techniques were developed. Among the mathematicians who worked with n-widths of sets of analytic functions, we highlight Babenko, Fisher, Micchelli, Taikov and Tichomirov, who obtained estimates for sets of analytic functions in Hardy spaces in [1,4,29,30]. Among the different tools used, we can emphasize the properties of the class of Blaschke products of degree m or less. In 1980, Melkman in [21] obtained estimates for a set of entire functions in $C[-T,T]$, using other technique.

Observing the historical evolution of the study of n-widths, it is possible to note that it has been an usual practice to use different techniques in proofs of lower and upper bounds of n-widths of Sobolev classes and of sets of analytic or entire functions.

One of the goals of this paper, in this sense, is to give an unified treatment for n-widths of sets of functions determined by multiplier operators, through of Theorems 3 and 4. Among the tools utilized in the proofs of these two theorems, the main is Theorem 1, which provides estimates for Levy Means of norms defined on $\mathbb{R}^n$. In the proof of this theorem we use, among other tools, the Khintchine inequality and a probabilistic result of Kwapień (Lemma 1). We also use other tools beyond Levy Means, among which we can highlight a result of Pajor and Tomczak-Jaegermann (Theorem 2).

In the first part, upper and lower bounds are established for n-widths of general multiplier operators. In the second part, we apply these results in the study of estimates for n-widths of sets of finitely and infinitely differentiable functions on $\mathbb{T}^d$. We show, in particular, that in various important situations the estimates are order sharp.

All the proofs in this paper will be proved only for the operators of type Λ. In Remark 4 we explain as the results can be obtained for the operators of type Λ_*.

Consider two Banach spaces X and Y. The norm of X will be denoted by $\|\cdot\|$ or $\|\cdot\|_X$ and the closed unit ball $\{x \in X : \|x\| \le 1\}$ by B_X. We begin by recalling well-known definitions. Let A be a compact, convex and centrally symmetric subset of X. The Kolmogorov n-width of A in X is defined by

$$d_n(A,X) = \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|_X,$$