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Stability of ball proximinality

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Abstract

In this paper, we show that if E is an order continuous Köthe function space and Y is a separable subspace of X, then $E(Y)$ is ball proximinal in $E(X)$ if and only if Y is ball proximinal in X. As a consequence, $E(Y)$ is proximinal in $E(X)$ if and only if Y is proximinal in X. This solves an open problem of Bandyopadhyay, Lin and Rao. It is also shown that if E is a Banach lattice with a 1-unconditional basis and for each n, Y_n is a subspace of X_n, then $(\oplus Y_n)_E$ is ball proximinal in $(\oplus X_n)_E$ if and only if each Y_n is ball proximinal in X_n.

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1. Introduction

For any real Banach space X, we denote by B_X and S_X, respectively, the closed unit ball and unit sphere of X. Let C be a nonempty closed convex subset of X. For any $x \in X$ and $\delta > 0$,

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$P_C(x)$ and $P_C(x,\delta)$ denote the sets

$$P_C(x) = \{z \in C : \|x - z\| = d(x, C)\},$$
$$P_C(x,\delta) = \{y \in C : \|x - y\| < d(x, C) + \delta\}.$$

C is said to be *proximinal* if, for every x in X, the set $P_C(x) \neq \emptyset$. A proximinal subset C is said to be *strongly proximinal* if for any $\varepsilon > 0$ and any $x \in X$, there exists $\delta > 0$, such that $P_C(x,\delta) \subset P_C(x) + \varepsilon B_X$. Let Y be a subspace of X. Here subspaces of X always mean closed vector subspaces. Y is said to be *ball proximinal* (respectively, *strongly ball proximinal*) if the closed unit ball B_Y is proximinal (respectively, strongly proximinal) [1]. It is easy to see that if Y is a (strongly) ball proximinal subspace of X, then Y is a (strongly) proximinal subspace of X. In [8], Saidi showed for any nonreflexive space Y there is a Banach space X such that Y is isometrically isomorphic to a subspace Z of X such that Z is proximinal in X; but not ball proximinal in X. Bandyopadhyay, Lin, and Rao [1, Example 3.3] showed that the space Z is strongly proximinal in X.

Let (Ω, Σ, μ) be a measure space. A Köthe function space E is a Banach space of measurable real-valued functions that satisfies the following conditions:

(1) For any finite measurable set A, $\chi_A \in E$.
(2) If $g \in E$ and f is a measurable function such that $|f(\omega)| \leq |g(\omega)|$ for almost all ω, then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

E is said to be strictly monotone if for any $0 < f < g$, $\|f\|_E < \|g\|_E$. E is said to be uniformly monotone if for any ϵ, there is $\delta > 0$ such that if $\|f\|_E = 1$ and $\|g\|_E > \delta$, then $\big\| |f| + |g| \big\| > 1 + \epsilon$. It is easy to see that for any $1 \leq p < \infty$, L_p is uniformly monotone. E is said to be order continuous, if for any decreasing sequence (f_n) in E that converges to 0 for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \|f\|_E = 0.$$

Let X be a Banach space. The Köthe–Bochner function space $E(X)$ is the collection of X-valued measurable functions F such that $\|F(\cdot)\|_X \in E$. The norm of F is defined by

$$\|F\|_{E(X)} = \Big\| \|F(\cdot)\|_X \Big\|_E.$$

Several papers [3,6,7] have been devoted to studying the following problem:

Question 1. Let E be a Köthe function space and Y a subspace of a Banach space X. Is $E(Y)$ proximinal in $E(X)$ if Y is proximinal in X?

Mendoza proved the following:

(1) Let X be a Banach space, let Y be a closed separable subspace of X, and let $1 \leq p \leq \infty$. $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ if and only if Y is proximinal in X.
(2) There exists a Banach space such that X has a proximinal subspace Y such that $L_p([0, 1], Y)$ is not proximinal in $L_p([0, 1], X)$ for any $1 \leq p \leq \infty$.

Bandyopadhyay, Lin, and Rao [1] showed if (X_n) and (Y_n) are two sequences of Banach spaces such that Y_n is a subspace of X_n that is ball proximinal in X_n for each n, then $(\oplus Y_n)_{c_0}$ is ball proximinal in $(\oplus X_n)_{c_0}$. They asked

Question 2. Let E be a Köthe function space and X a Banach space. Is $E(Y)$ ball proximinal in $E(X)$ if Y is ball proximinal in X?