Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 186 (2014) 28–32

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Two measures on Cantor sets

Gökalp Alpan, Alexander Goncharov*

Bilkent University, 06800, Ankara, Turkey

Received 22 April 2014; received in revised form 12 June 2014; accepted 9 July 2014
Available online 23 July 2014

Communicated by Vilmos Totik

Abstract

We give an example of Cantor-type set for which its equilibrium measure and the corresponding Hausdorff measure are mutually absolutely continuous. Also we show that these two measures are regular in the Stahl–Totik sense.

MSC: 30C85; 31A15; 28A78; 28A80

Keywords: Harmonic measures; Hausdorff measures; Cantor sets

1. Introduction

The relation between the α dimensional Hausdorff measure Λ_α and the harmonic measure ω on a finitely connected domain Ω is understood well. Due to Makarov [5], we know that, for a simply connected domain, $\dim \omega = 1$ where $\dim \omega := \inf\{\alpha : \omega \perp \Lambda_\alpha\}$. Pommerenke [9] gives a full characterization of parts of $\partial\Omega$ where ω is absolutely continuous or singular with respect to a linear Hausdorff measure. Later similar facts were obtained for finitely connected domains. In the infinitely connected case there are only particular results. Model example here is $\Omega = \overline{\mathbb{C}} \setminus K$ for a Cantor-type set K. For all such cases we have $\Lambda_{\alpha_K} \perp \omega$ on K, because of the strict inequality $\dim \omega < \alpha_K$ (see, e.g. [1,6,7,12,14]), where α_K stands for the Hausdorff dimension of K.

* Corresponding author.
E-mail addresses: gokalp@fen.bilkent.edu.tr (G. Alpan), goncha@fen.bilkent.edu.tr (A. Goncharov).

http://dx.doi.org/10.1016/j.jat.2014.07.003

These results motivate the problem to find a Cantor set for which its harmonic measure and the corresponding Hausdorff measure are not mutually singular.

Recall that, for a dimension function h, a set $E \subset \mathbb{C}$ is an h-set if $0 < \Lambda_h(E) < \infty$ where Λ_h is the Hausdorff measure corresponding to the function h. We consider Cantor-type sets $K(\gamma)$ introduced in [3]. In Section 2 we present a function h that makes $K(\gamma)$ an h-set. In Section 3 we show that Λ_h and ω are mutually absolutely continuous for $K(\gamma)$. In the last section we prove that these two measures are regular in the Stahl–Totik sense.

We will denote by log the natural logarithm, and $Cap(\cdot)$ stands for the logarithmic capacity.

2. Dimension function of $K(\gamma)$

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is called a dimension function if it is increasing, continuous and $h(0) = 0$. Given set $E \subset \mathbb{C}$, its h-Hausdorff measure is defined as

$$\Lambda_h(E) = \lim_{\delta\to 0} \inf \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \le \delta \right\}, \tag{2.1}$$

where $B(z, r)$ is the open ball of radius r centered at z.

For the convenience of the reader we repeat the relevant material from [3]. Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s \le \frac{1}{32}$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s} \cdot (P_{2^s} + r_s)$ for $s \in \mathbb{N}$. Consider the set

$$E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \le 0\} = \cup_{j=1}^{2^s} I_{j,s}.$$

The sth level *basic* intervals $I_{j,s}$ with lengths $l_{j,s}$ are disjoint and $\max_{1\le j\le 2^s} l_{j,s} \to 0$ as $s \to \infty$. Since $E_{s+1} \subset E_s$, we have a Cantor-type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$. The set $K(\gamma)$ is non-polar if and only if $\sum_{s=1}^{\infty} 2^{-s} \log \frac{1}{\gamma_s} < \infty$. In this paper we make the assumption

$$\sum_{s=1}^{\infty} \gamma_s < \infty. \tag{2.2}$$

Let $M := 1 + \exp\left(16 \sum_{s=1}^{\infty} \gamma_s\right)$, so $M > 2$, and $\delta_s := \gamma_1\gamma_2 \ldots \gamma_s$. By Lemma 6 in [3],

$$\delta_s < l_{j,s} < M \cdot \delta_s \quad \text{for } 1 \le j \le 2^s. \tag{2.3}$$

We construct a dimension function for $K(\gamma)$, following Nevanlinna [8]. Let $\eta(\delta_s) = s$ for $s \in \mathbb{Z}_+$ with $\delta_0 := 1$. We define $\eta(t)$ for (δ_{s+1}, δ_s) by

$$\eta(t) = s + \frac{\log \frac{\delta_s}{t}}{\log \frac{\delta_s}{\delta_{s+1}}}.$$

This makes η continuous and monotonically decreasing on $(0, 1]$. In addition, we have $\lim_{t\to 0} \eta(t) = \infty$. Also observe that, for the derivative of η on (δ_{s+1}, δ_s), we have

$$\frac{d\eta}{dt} = \frac{-1}{t \log \frac{1}{\gamma_{s+1}}} \ge \frac{-1}{t \log 32} \quad \text{and} \quad \frac{d\eta}{d \log t} \ge \frac{-1}{\log 32}.$$

Define $h(t) = 2^{-\eta(t)}$ for $0 < t \le 1$ and $h(t) = 1$ for $t > 1$. Then h is a dimension function with $h(\delta_s) = 2^{-s}$ and

$$\frac{d \log h}{d \log t} < \frac{\log 2}{\log 32} < 1.$$