Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 187 (2014) 1–17

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Local behavior of the equilibrium measure under an external field non differentiable at a point

J.F. Sánchez-Lara

Departamento de Matemática Aplicada, Universidad de Granada, Campus de la Cartuja s/n, 18071 Granada, Spain

Received 29 August 2013; received in revised form 23 June 2014; accepted 24 July 2014
Available online 7 August 2014

Communicated by Vilmos Totik

Abstract

The equilibrium measure, μ, associated with an external field, φ, defined on $\Sigma \subset \mathbb{R}$ and such that there exists a point x_0 and an open interval, I, with $x_0 \in I \subset \Sigma$, φ being continuous in I but analytic in $I \setminus \{x_0\}$, is studied. It is shown that the main asymptotic term of the derivatives of φ at both sides of x_0 determines the local behavior of $\mathrm{Supp}(\mu)$ and the local behavior of the density associated with μ. Some results related to a "forbidden region" for $\mathrm{Supp}(\mu)$ are obtained. Also the behavior of the density associated with μ is obtained when x_0 is contained in the interior of the support.

MSC: primary 31A15; secondary 30E20; 30E25

Keywords: Logarithmic potentials

1. Introduction and statement of results

The theory of logarithmic potentials has shown to be a key tool to describe a very wide class of mathematical problems related with families of nodes satisfying some extremal property especially when the amount of these nodes grows large. For instance, it describes the asymptotic behavior of the distribution of zeros of orthogonal polynomials. But also it is one of the main tools

E-mail address: jslara@ugr.es.

http://dx.doi.org/10.1016/j.jat.2014.07.011
0021-9045/

in the analysis of orthogonal polynomials via the Riemann–Hilbert problem, which is a richer description of the orthogonal polynomials than the limit distribution of the zeros. Furthermore, logarithmic potentials are intimately related with techniques in approximation theory. For instance, the limit distribution of nodes of a wide class of problems such as quadratures, uniform approximation, Padé and rational approximation, etc., is controlled by an equilibrium measure under an external field. But this is not the only type of problems where the equilibrium with logarithmic potentials appears. For instance, the study of the contact points in 2D between an elastic material and a rigid punch is described also by an equilibrium measure with logarithmic potentials.

The main step in solving the equilibrium problem is to find the support, $\mathrm{Supp}(\mu)$, of the equilibrium measure, μ. The nature of $\mathrm{Supp}(\mu)$ depends strongly on the nature of the external field considered. For instance there are a few conditions on φ that guarantee that $\mathrm{Supp}(\mu)$ is one interval or a finite family of intervals. But it is not too clear a priori which points can belong to $\mathrm{Supp}(\mu)$, probably the only exception being the minimum of φ, which is expected to be in $\mathrm{Supp}(\mu)$ if φ is continuous at such point. If the external field has a jump discontinuity at some point, x_0, then x_0 does not belong to the interior of $\mathrm{Supp}(\mu)$ (which can be proven arguing that if this were not the case, the function given by the total potential, $V_\mu+\varphi$, for $x \neq x_0$ and the equilibrium constant for $x = x_0$, would be constant in a neighborhood of x_0 by Theorem I.4.8 in [8] and the lower semi-continuity, but on the other side this function cannot be continuous at x_0 in the fine topology [8, Section I.5], which leads to a contradiction). Thus a natural question is what happens if φ is continuous but non differentiable at a point.

Let us consider $\Sigma \subset \mathbb{R}$ and without loss of generality let us also take $x_0 = 0$ being an interior point of Σ, so there exists an open interval $I \subset \Sigma$ with $0 \in I$. The external field to be considered is such that it can be expressed in I as

$$\varphi(x) = \begin{cases} |x|^\alpha f(x) & \text{for } x < 0,\ x \in I, \\ x^\beta g(x) & \text{for } 0 \le x,\ x \in I, \end{cases} \tag{1}$$

with $\alpha, \beta \in (0, +\infty)$, f and g real analytic functions on I such that $f(0) \neq 0$ and $g(0) \neq 0$. We assume that $\min\{\alpha, \beta\} \le 1$; strict inequality guaranties an unbounded jump of φ' at 0, while the case of φ' with a bounded jump corresponds with equality. If $\alpha = \beta = 1$, then we also need to assume that $f(0) + g(0) \neq 0$.

Remark 1. For simplicity, functions f and g are assumed to be analytic, but most arguments work if they are of class $\mathcal{C}^{1+\varepsilon}$ (differentiable functions with derivatives satisfying a Hölder-continuity with exponent ε). Also, a more general type of external fields can be considered, for instance if the external field in a neighborhood of 0 is the sum of φ with a function of class $\mathcal{C}^{1+\varepsilon}$, and the following results remain valid.

With these assumptions, take $\gamma = \min\{\alpha, \beta\} \in (0, 1]$ and

$$\Delta = \lim_{\varepsilon \to 0^+} \frac{\varphi'(\varepsilon) - \varphi'(-\varepsilon)}{\varepsilon^{\gamma-1}},$$

which exists as a bounded and nonzero value. The main result is the following.

Theorem 1. *Consider an external field that in a neighborhood of* 0 *can be expressed as* (1) *and* μ *the associated equilibrium measure. If* $\Delta < 0$*, then* $0 \notin \mathrm{Supp}(\mu)$.

At this point, this result needs some comments.