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Markov–Bernstein and Landau–Kolmogorov type inequalities in several variables for the Hermite and closely connected measures

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Abstract

This paper is devoted to Markov–Bernstein and Landau–Kolmogorov type inequalities in several variables in L^2 norm. The measures which are used, are the Hermite measure and some closely connected measures in such a way that the partial derivatives of the orthogonal polynomials are also orthogonal. These orthogonal polynomials in several variables are built by tensor product of the orthogonal polynomials in one variable. These inequalities are obtained by using a variational method and they involve the square norms of a polynomial p and at most those of the partial derivatives of order 2 of p.

It is also proved that the sequences of polynomials in several variables, orthogonal with respect to the Hermite and Laguerre–Sonin measures, are closed in a certain Sobolev space regarding its natural inner product and the inner product linked to the bilinear functional which is used in the variational method. This result shows us that the given Landau–Kolmogorov type inequalities are satisfied not only for polynomials, but also for functions belonging to this Sobolev space.

MSC: 33C45; 33C50; 26D05

Keywords: Markov–Bernstein inequalities; Landau–Kolmogorov type inequalities; L^2 norm; Hermite measure; Laguerre–Sonin measure; Jacobi measure; Orthogonal polynomials; Several variables; Variational method; Sobolev spaces

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1. Introduction

In this paper we want to present some inequalities in several variables. To make more understandable the problem of Markov–Bernstein and Landau–Kolmogorov type inequalities in several variables, it is simpler to start with some recalls in one variable. The notations actually are simpler in this case. We will give the most general definition of these inequalities in one variable, as well as the results which concern them, obtained by several authors. An approach of this problem in one variable consists of the study of the positivity of a certain symmetric bilinear functional a_λ. This tool was introduced in several previous papers (see [9,10]).

Let us define this functional.

Let $L^2(\Omega;\mu_m), m = 0,\dots,N$, be the Hilbert space of square integrable real functions on the open set $\Omega \subset \mathbb{R}$ for the positive Borel measure μ_m supported on Ω with finite moments $(\int_\Omega x^k d\mu_m(x))$ of any order $k \in \mathbb{N}$. Its inner product is defined by

$$(f,g)_{L^2(\Omega;\mu_m)} = \int_\Omega f(x)g(x)d\mu_m(x), \quad \forall f,g \in L^2(\Omega;\mu_m),$$

and the norm is

$$\|f\|_{L^2(\Omega;\mu_m)} = ((f,f)_{L^2(\Omega;\mu_m)})^{\frac{1}{2}}.$$

Let $H^N(\Omega;\mu_0,\mu_1,\dots,\mu_N)$ be the Sobolev space defined by

$$H^N(\Omega;\mu_0,\mu_1,\dots,\mu_N) = \left\{f \,|\, f^{(m)} \in L^2(\Omega;\mu_m), m = 0,\dots,N\right\}$$

where the derivative $f^{(m)}$ of order m is taken in the distributional sense. The classical inner product on this space is given by

$$(f,g)_{H^N(\Omega;\mu_0,\mu_1,\dots,\mu_N)} = \sum_{m=0}^{N}(f^{(m)},g^{(m)})_{L^2(\Omega;\mu_m)}, \quad \forall f,g \in H^N(\Omega;\mu_0,\mu_1,\dots,\mu_N),$$

and the corresponding norm is

$$\|f\|_{H^N(\Omega;\mu_0,\dots,\mu_N)} = ((f,f)_{H^N(\Omega;\mu_0,\dots,\mu_N)})^{\frac{1}{2}}.$$

Now, we consider the following symmetric bilinear functional a_λ

$$a_\lambda : H^N(\Omega;\mu_0,\dots,\mu_N) \times H^N(\Omega;\mu_0,\dots,\mu_N) \to \mathbb{R}$$

defined by

$$a_\lambda(f,g) = \sum_{m=0}^{N} \lambda_m (f^{(m)},g^{(m)})_{L^2(\Omega;\mu_m)},$$

$\forall f$ and $g \in H^N(\Omega;\mu_0,\dots,\mu_N)$. $\lambda_m, m = 0,\dots,N$, are $N+1$ fixed real numbers such that $\lambda_0 = 1$ and $\lambda_N \neq 0$. λ will denote the vector of $\mathbb{R}^N$ whose entries are $\lambda_m, m = 1,\dots,N$.

To obtain the concerned inequalities, we have to look for $\lambda \in \mathbb{R}^N$ such that $a_\lambda(p,p) \geqslant 0, \forall p \in \mathcal{P}_n$. $\mathcal{P}_n$ is the vector space of real polynomials in one variable of degree at most equal to n.

When N is fixed, the results of the positivity of $a_\lambda(p,p)$ can be separated in two groups. The first group corresponds to the case where $\lambda_m, m = 1,\dots,N-1$, are fixed equal to 0. Then, we