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Spaces of generalized smoothness in the critical case: Optimal embeddings, continuity envelopes and approximation numbers

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Abstract

We study necessary and sufficient conditions for embeddings of Besov spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ when $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$. A borderline situation, corresponding to the limiting situation in the classical case, is included and give new results. In particular, we characterize optimal embeddings for B-spaces.

As immediate applications of our results we obtain continuity envelopes and give upper and lower estimates for approximation numbers for the related embeddings.

We also consider the analogous results for the Triebel–Lizorkin spaces of generalized smoothness $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$.

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1. Introduction

Spaces of generalized smoothness have been studied by several authors, including different approaches. We follow the general Fourier-analytical approach as presented in [14]. There one can find more details and some history on these spaces.

The reason for the revived interest in the study of Besov and Triebel–Lizorkin spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ is its connection with applications for pseudo-differential operators (as generators of sub-Markovian semi-groups), cf. [14].

The aim of this paper is to complete the study of [23] by extending the results obtained in [34,33] to Besov and Triebel–Lizorkin spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$, where a borderline situation, corresponding to the limiting situation in the classical case, is considered and gives new results. We also give examples which yield results that are not covered by the previous references. Moreover, we obtain new estimates for approximation numbers.

In the present paper (cf. Theorem 3.2 below), we give necessary and sufficient conditions for embeddings of $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ when $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$, i.e., we show that

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{1.1}$$

if, and only if,

$$\sup_{M\geq 0}\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\sum_{k=M}^{\infty}\tau_k^{-q'}\right)^{\frac{1}{q'}} < \infty$$

(with the usual modification if $r = \infty$ and/or $q' = \infty$), provided that $0 < p \leq \infty$, $0 < q \leq r \leq \infty$, $\boldsymbol{\sigma} = \{\sigma_j\}_{j\in\mathbb{N}_0}$ and $\boldsymbol{N} = \{N_j\}_{j\in\mathbb{N}_0}$ are admissible sequences, the latter satisfying $\underline{N}_1 > 1$, and $\mu \in \mathcal{L}_r$ (see Section 3 for precise definitions).

Furthermore, (cf. Corollary 3.4), when $q > 1$ and $r \in [q,\infty]$, the embedding (1.1) with $\mu = \lambda_{qr}$,

$$\lambda_{qr}(t) = \left(\Lambda(t^{-1})t^{n/p}\right)^{\frac{q'}{r}}\left(\int_0^t\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad t \in (0, N_0^{-1}],$$

where Λ is an admissible function such that $\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, with equivalence constants independent of j, is sharp with respect to the parameter μ, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ in (1.1) and the space $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n)$ (i.e., the target space in (1.1) with $\mu = \lambda_{qr}$) satisfy $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$. The embedding with $r = q$ and $\mu = \lambda_{qr} = \lambda_{qq}$ is optimal (i.e., it is the best possible embedding among all the embeddings considered in (1.1)). The case $0 < q \leq 1$ is considered in Remark 3.5.

We also consider the analogous results for the Triebel–Lizorkin spaces of generalized smoothness $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ (cf. Theorem 3.6, Corollary 3.8 and Remark 3.9 below).

Concerning applications, we compute continuity envelopes $\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}^X_{\mathsf{C}}(t), u^X_{\mathsf{C}}\right)$ (cf. Definition 4.1 below), which are closely related with sharp embeddings, where $\mathcal{E}^X_{\mathsf{C}}(t) := \sup_{\|f|X\|\leq 1}\frac{\omega(f,t)}{t}$, $t > 0$, together with some fine index u^X_{C}; here $\omega(f,t)$ stands for the modulus of continuity.