Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 181 (2014) 1–5
Journal of Approximation Theory
www.elsevier.com/locate/jat

Notes

A note on strong asymptotics of weighted Chebyshev polynomials

András Kroó*,[1]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary
Budapest University of Technology and Economics, Department of Analysis, Budapest, Hungary

Received 20 July 2013; received in revised form 30 December 2013; accepted 19 January 2014
Available online 31 January 2014

Communicated by Vilmos Totik

Abstract

Denote by $T_n(., w)$ the nth degree Chebyshev polynomial with respect to the positive weight w. It is shown in this note that if w is C^{1+} then $T_n(\cos\phi, w) = \Re\{e^{-in\phi}\pi^2(e^{i\phi})\} + o(1)$ where $\pi(z)$ stands for the Szegő function corresponding to w. This extends an earlier result proved by Kroó and Peherstorfer who verified this asymptotic relation for $C^{2+\alpha}$, $\alpha > 0$ weights.

Keywords: Weighted Chebyshev polynomials; Asymptotics; Strong unicity constant

Let w be a positive continuous weight function on $[-1, 1]$ and denote by

$$T_n(x, w) = a_n x^n + \cdots + a_0, \quad a_n > 0$$

the normalized Chebyshev polynomial of degree n with respect to w uniquely defined by

$$1 = \|T_n(., w)w\| \le \|(a_n x^n + b_{n-1}x^{n-1} + \cdots + b_0)w(x)\|, \quad \forall b_{n-1}, \ldots, b_0 \in \mathbb{R}.$$

Here $\|\cdots\|$ stands for the usual supremum norm on $[-1, 1]$. By the classical Chebyshev alternation theorem $T_n(., w)$ is characterized by the property that function $T_n(., w)w$ equioscillates $n+1$ times on $[-1, 1]$ between 1 and -1.

* Correspondence to: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary.
E-mail address: kroo.andras@renyi.mta.hu.

[1] Written during the author's visit as Chair of Excellence at the Carlos III University of Madrid.

0021-9045/$ - see front matter
http://dx.doi.org/10.1016/j.jat.2014.01.005

In case when $w = \frac{1}{\rho_m}$ with ρ_m a polynomial of degree $m < n$ positive on $[-1, 1]$ the extremal polynomial $T_n(., w)$ can be found explicitly; this result goes back to Chebyshev (see [1]). Namely, let g_m be the well-known Fejer–Riesz representation of ρ_m, that is g_m is the unique polynomial of degree m with real coefficients and all zeros in $|z| > 1$ such that $|g_m(e^{i\phi})|^2 = \rho_m(\cos\phi)$, $g_m(0) > 0$. Then with $z = e^{i\phi}$, $x = \cos\phi$ we have

$$T_n(x, 1/\rho_m) = \Re\{z^{-n} g_m^2(z)\}. \tag{1}$$

Clearly, $T_n(., 1/\rho_m)/\rho_m$ equioscillates $n + 1$ times on $[-1, 1]$ between 1 and -1. Since

$$\Re\{z^{-n} g_m^2(z)\}^2 + \Im\{z^{-n} g_m^2(z)\}^2 = \rho_m^2, \quad z = e^{i\phi} \tag{2}$$

the $n + 1$ equioscillations of $T_n(., 1/\rho_m)/\rho_m$ are zeros of $\Im\{z^{-n} g_m^2(z)\}^2$.

Finding asymptotic representation for the weighted Chebyshev polynomials has been one of the basic problems in Approximation Theory for many years. The nth root asymptotic of the weighted Chebyshev polynomials was given by Fekete and Walsh [3]; strong asymptotics outside $[-1, 1]$ was found by Lubinsky and Saff (see [7,6]).

The first general result on strong asymptotics for weighted Chebyshev polynomials on $[-1, 1]$ was given in [4] where it was shown that if w is of the smoothness class $C^{2+\alpha}$, $\alpha > 0$ (that is the *second derivative* of $w(\cos\phi)$ is Lipα) then $T_n(x, w) = \Re\{z^{-n}\pi^2(z)\} + o(1)$, $z = e^{i\phi}$, $x = \cos\phi$ uniformly for $\phi \in [0, \pi]$. Here $\pi(z)$ stand for the classical Szegő function corresponding to the weight w, that is $\pi(z)$ is the analytic and nonzero function on $|z| < 1$ satisfying the boundary condition $|\pi(z)| = w(\cos\phi)^{-1/2}$, $z = e^{i\phi}$.

In this note we will show that this result can be improved substantially by relaxing the smoothness requirement on the weight. In fact we will prove that the above asymptotic relation holds for weights whose *first derivatives are of the Dini–Lipschitz class*. For the positive and differentiable on $[-1, 1]$ weight w denote by $\omega_1(h)$ the modulus of continuity of the *first derivative* of $w(\cos\phi)$. We shall say that $w \in C^{1+}$ if $\omega_1(h)$ satisfies the Dini–Lipschitz property

$$\omega_1\left(\frac{1}{n}\right) = o\left(\frac{1}{\log n}\right), \quad n \to \infty. \tag{3}$$

Theorem. *Let $w \in C^{1+}$ be a positive weight on $[-1, 1]$. Then as $n \to \infty$*

$$T_n(\cos\phi, w) = \Re\{e^{-in\phi}\pi^2(e^{i\phi})\} + o(1), \tag{4}$$

uniformly for $\phi \in [0, \pi]$.

Denote by P_n the space of polynomials with real coefficients of degree at most $n - 1$. Given $f \in C[-1, 1]$ having 0 as its best approximant out of P_n, let $\gamma_n(f, w)$ be the nth weighted *strong unicity constant* of f defined as

$$\gamma_n(f, w) := \sup_{p_n \in P_n \setminus \{0\}} \frac{\|wp_n\|}{\|w(f - p_n)\| - \|wf\|}. \tag{5}$$

It is shown in [4], Lemma 3.2 that given positive weights $w_1, w_2 \in C[-1, 1]$ we have that

$$\|T_n(., w_1) - T_n(., w_2)\| \le c\gamma_n(T_n(., w_2), w_2)\|w_1 - w_2\| \tag{6}$$

where the positive constant c depends only on the minima and maxima of weights w_1, w_2 on $[-1, 1]$.