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Splines on the Alfeld split of a simplex and type A root systems

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Abstract

Alfeld (1984) introduced a tetrahedral analog $AS(\Delta_3)$ of the Clough–Tocher split of a triangle. A formula for the dimension of the spline space $C_k^r(AS(\Delta_n))$ is conjectured in Foucart and Sorokina (2013). We prove that the graded module of C^r-splines on the cone over $AS(\Delta_n)$ is isomorphic to the module $D^{r+1}(A_n)$ of multiderivations on the type A_n Coxeter arrangement. A theorem of Terao shows that the module of multiderivations of a Coxeter arrangement is free and gives an explicit basis. As a consequence the conjectured formula holds.

Keywords: Spline; Hyperplane arrangement; Dimension formula

1. Introduction

The Alfeld split $AS(\Delta_n)$ of the n-simplex Δ_n is obtained by adding a single central interior vertex, and then coning over the boundary of Δ_n. Hence for $i > 0$, every $i + 1$-face of $AS(\Delta_n)$ corresponds to an i-face of $\partial(\Delta_n)$, and $AS(\Delta_n)$ has a single interior vertex. This construction was introduced in [1] as a tetrahedral analog of the Clough–Tocher split of a triangle, and [6] makes the following.

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Conjecture 1.1 (*Foucart–Sorokina*)**.** *The dimension of the space $C^r_k(AS(\Delta_n))$ of splines of degree $\leq k$ in n-variables over the Alfeld split of Δ_n is given by*

$$\dim C^r_k(AS(\Delta_n)) = \binom{k+n}{n} + \begin{cases} n\binom{k+n-\frac{(r+1)(n+1)}{2}}{n}, & \textit{if } r \textit{ is odd} \\ \binom{k+n-1-\frac{r(n+1)}{2}}{n} + \cdots + \binom{k-\frac{r(n+1)}{2}}{n}, \\ \quad \textit{if } r \textit{ is even.} \end{cases}$$

In this brief note, we prove the conjecture. The proof hinges on a connection between splines and the theory of hyperplane arrangements.

In [2], Billera showed that splines on a simplicial complex Δ can be obtained as the top homology module of a chain complex, and used this to solve a longstanding conjecture of Gil Strang. Billera–Rose show in [3] that by forming the cone $\hat{\Delta} \subseteq \mathbb{R}^{n+1}$ over $\Delta \subseteq \mathbb{R}^n$, commutative algebra can be used to study the dimension of $C^r_k(\Delta)$: $C^r(\hat{\Delta})$ is a graded module over the polynomial ring $S = \mathbb{R}[x_0, \ldots, x_n]$, and the dimension of $C^r_k(\Delta)$ is the dimension of the kth graded piece of $C^r(\hat{\Delta})$.

A spectral sequence argument in [10] shows that $C^r(\hat{\Delta})$ is a free module iff all the lower homology modules of a certain chain complex vanish, and Corollary 4.12 of [10] shows that in this case the dimension of $C^r_k(\Delta)$ is determined entirely by local data. In [7], the algebraic geometry of fat points in projective space is used to study the dimension of quotients of ideals generated by powers of linear forms. Such modules appear in the chain complex whose top homology module is $C^r(\hat{\Delta})$. In Section 2 we give more details about these methods, and in Section 3 we show Conjecture 1.1 follows from a theorem of Terao [12]. The case $r = 1$ was recently proved in [8].

2. Algebraic preliminaries

Throughout the paper, our references are de Boor [4] for splines and Eisenbud [5] for commutative algebra.

Definition 2.1 (*[10]*)**.** For a simplicial complex $\Delta \subseteq \mathbb{R}^n$, let $\mathcal{R}/\mathcal{J}(\Delta)$ be the complex of $S = \mathbb{R}[x_0, \ldots, x_n]$ modules, with differential ∂_i the usual boundary operator in relative (modulo boundary) homology.

$$0 \longrightarrow \bigoplus_{\sigma \in \Delta_n} S \xrightarrow{\partial_n} \bigoplus_{\tau \in \Delta^0_{n-1}} S/I_\tau \xrightarrow{\partial_{n-1}} \bigoplus_{\psi \in \Delta^0_{n-2}} S/I_\psi \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \bigoplus_{v \in \Delta^0_0} S/I_v \longrightarrow 0,$$

where for an interior i-face $\gamma \in \Delta^0_i$, we define

$$I_\gamma = \langle l^{r+1}_{\hat{\tau}} \mid \hat{\gamma} \subseteq \hat{\tau} \in \hat{\Delta}_{n-1} \rangle.$$

This differs from the complex in [2]: the ideal I_γ is generated by $r + 1^{st}$ powers of (homogenizations of) linear forms whose vanishing defines the hyperplanes τ which contain γ. As with the complex in [2], the top homology module of $\mathcal{R}/\mathcal{J}(\Delta)$ computes splines of smoothness r on $\hat{\Delta}$. Theorem 4.10 of [10] shows that if Δ is a topological n-ball, then the module $C^r(\hat{\Delta})$ is free iff $H_i(\mathcal{R}/\mathcal{J}(\Delta)) = 0$ for all $i < n$. Computing the dimension of the modules which appear in the complex $\mathcal{R}/\mathcal{J}(\Delta)$ is nontrivial as soon as $n \geq 3$. Indeed, for ideals generated by powers of linear forms in three variables, no answer is known, even if the forms are in general position. This is due