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Sampling inequalities in Sobolev spaces

Rémi Arcangéli[a], Juan José Torrens[b,*]

[a] Route de Barat, 31160 Arbas, France

[b] Departamento de Ingeniería Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía, 31006 Pamplona, Spain

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Abstract

Sampling inequalities in the Sobolev space $W^{r,p}(\Omega)$, where Ω is a domain of $\mathbb{R}^n$, are defined as relations like

$$|u|_{l,q,\Omega} \le C \left(d^{r-l-n(1/p-1/q)} |u|_{r,p,\Omega} + d^{n/q-l} \left(\sum_{a\in A} |u(a)|^p \right)^{1/p} \right), \; l \le \ell,$$

for suitable values of r, p, q and ℓ. In this statement, u denotes a function in $W^{r,p}(\Omega)$, A is a *discrete* set in $\overline{\Omega}$ and $d = \sup_{x\in\Omega} \inf_{a\in A} |x-a|$.

The structure of the sampling inequalities in spaces $W^{r,p}(\Omega)$ is analysed, their role in the study of the interpolation error by spline functions is recalled, and the analogy between these inequalities and those relative to intermediate semi-norms in spaces $W^{r,p}(\Omega)$ (cf., for example, Adams and Fournier (2003) [1, Theorem 5.2–(1)]) is shown.

Keywords: Sampling inequality; Sobolev space; Spline; Bound for intermediate semi-norms

* Corresponding author.
E-mail addresses: arcangeli.remi@wanadoo.fr (R. Arcangéli), jjtorrens@unavarra.es (J.J. Torrens).

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1. Introduction

1.1. Background

The concept of *sampling inequality* was introduced by Rieger in his Ph.D. Thesis (cf. [14]; see also Rieger et al. [15]) in order to systematize a new kind of bounds developed, in their origins, to cope with error estimates, using Sobolev norms, in scattered data interpolation. Due to their increasing number of applications in areas like approximation theory, machine learning or meshless methods for PDE, the research on sampling inequalities has received an increasing attention. In our opinion, some milestones in this field previous to Rieger's dissertation are the pioneering work of Duchon [8] and the papers by Wendland and Rieger [20], Narcowich et al. [13], Madych [12] and Arcangéli et al. [3]. Actual lines of research on sampling inequalities and their applications include, for example, the extension of sampling inequalities to functions defined on Riemannian manifolds (cf. Hangelbroek et al. [10]) or to spaces of infinitely smooth functions (cf. Rieger and Zwicknagl [16,17]), the derivation of Sobolev-type error estimates for several interpolation and approximation methods (cf. Fuselier and Wright [9], Lee et al. [11]), or the development of meshless methods for the numerical solution of PDE (cf. Schräder and Wendland [18]).

In this paper, we shall restrict our study to the classic frame of Sobolev spaces. We have two objectives in mind: (a) to provide new insights on the structure of sampling inequalities and, in particular, on the discrete term, and (b) to show that sampling inequalities are strongly related to and allow the recovering of existing bounds for intermediate semi-norms. We shall later precise these goals. Before that, we introduce the notations used throughout this paper and formalize, in this context, the notion of sampling inequality.

1.2. Notations

For any $x \in \mathbb{R}$, we shall write $\lfloor x \rfloor$ and $\lceil x \rceil$ for the *floor* (or integer part) and *ceiling* of x, that is, the unique integers satisfying $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ and $\lceil x \rceil - 1 < x \le \lceil x \rceil$. The letter n will always stand for an integer belonging to $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ (by convention, $0 \in \mathbb{N}$). Likewise, we write $\mathbb{R}_+^*$ for the set of positive real numbers. The Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$.

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|} / \big(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}\big)$, $x_1, \dots, x_n$ being the generic independent variables in $\mathbb{R}^n$.

Let Ω be a nonempty open set in $\mathbb{R}^n$. For any $r \in \mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r,p}(\Omega)$ the usual Sobolev space defined by

$$W^{r,p}(\Omega) = \{\, v \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n,\ |\alpha| \le r,\ \partial^\alpha v \in L^p(\Omega) \,\}.$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense. The space $W^{r,p}(\Omega)$ is equipped with the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0, \dots, r\}$, and the norm $\|\cdot\|_{r,p,\Omega}$ given by

$$|v|_{j,p,\Omega} = \Bigg(\sum_{|\alpha|=j} \int_\Omega |\partial^\alpha v(x)|^p \, dx \Bigg)^{1/p} \quad \text{and} \quad \|v\|_{r,p,\Omega} = \Bigg(\sum_{j=0}^{r} |v|_{j,p,\Omega}^p \Bigg)^{1/p}.$$

We remark that $|\cdot|_{0,p,\Omega}$ is, in fact, the usual L^p-norm $\|\cdot\|_{L^p(\Omega)}$. For any $r \in \mathbb{R}_+^* \setminus \mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r,p}(\Omega)$ the Sobolev space of noninteger order r, formed