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Exceptional Charlier and Hermite orthogonal polynomials☆

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Abstract

Using Casorati determinants of Charlier polynomials $(c_n^a)_n$, we construct for each finite set F of positive integers a sequence of polynomials c_n^F, $n \in \sigma_F$, which are eigenfunctions of a second order difference operator, where σ_F is certain infinite set of nonnegative integers, $\sigma_F \subsetneq \mathbb{N}$. For suitable finite sets F (we call them admissible sets), we prove that the polynomials c_n^F, $n \in \sigma_F$, are actually exceptional Charlier polynomials; that is, in addition, they are orthogonal and complete with respect to a positive measure. By passing to the limit, we transform the Casorati determinant of Charlier polynomials into a Wronskian determinant of Hermite polynomials. For admissible sets, these Wronskian determinants turn out to be exceptional Hermite polynomials.

Keywords: Orthogonal polynomials; Exceptional orthogonal polynomial; Difference operators; Differential operators; Charlier polynomials; Hermite polynomials

1. Introduction

Exceptional orthogonal polynomials p_n, $n \in X \subsetneq \mathbb{N}$, are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order

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differential operator. They extend the classical families of Hermite, Laguerre and Jacobi. The last few years have seen a great deal of activity in the area of exceptional orthogonal polynomials (see, for instance, [9,17,18] (where the adjective exceptional for this topic was introduced), [19,20,22,28,30,31,35,36,38] and the references therein).

The most apparent difference between classical orthogonal polynomials and exceptional orthogonal polynomials is that the exceptional families have gaps in their degrees, in the sense that not all degrees are present in the sequence of polynomials (as it happens with the classical families) although they form a complete orthonormal set of the underlying L^2 space defined by the orthogonalizing positive measure. This means in particular that they are not covered by the hypotheses of Bochner's classification theorem [4]. Exceptional orthogonal polynomials have been applied to shape-invariant potentials [35], supersymmetric transformations [19], discrete quantum mechanics [31], mass-dependent potentials [28], and quasi-exact solvability [38].

In the same way, exceptional discrete orthogonal polynomials are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order difference operator, extending the discrete classical families of Charlier, Meixner, Krawtchouk and Hahn. As far as the author knows the only known example of what can be called exceptional Charlier polynomials appeared in [39]. If orthogonal discrete polynomials on nonuniform lattices and orthogonal q-polynomials are considered, then one should add [31–34] where exceptional Wilson, Racah, Askey–Wilson and q-Racah polynomials are considered.

The purpose of this paper (and the forthcoming ones) is to introduce a systematic way of constructing exceptional discrete orthogonal polynomials using the concept of dual families of polynomials (see [27]). One can then also construct examples of exceptional orthogonal polynomials by taking limits in some of the parameters in the same way as one goes from classical discrete polynomials to classical polynomials in the Askey tableau.

Definition 1.1. Given two sets of nonnegative integers $U, V \subset \mathbb{N}$, we say that the two sequences of polynomials $(p_u)_{u\in U}$, $(q_v)_{v\in V}$ are dual if there exist a couple of sequences of numbers $(\xi_u)_{u\in U}$, $(\zeta_v)_{v\in V}$ such that

$$\xi_u p_u(v) = \zeta_v q_v(u), \quad u \in U, v \in V. \tag{1.1}$$

Duality has shown to be a fruitful concept regarding discrete orthogonal polynomials, and its utility will be again manifest in the exceptional discrete polynomials world. Indeed, it turns out that duality interchanges exceptional discrete orthogonal polynomials with the so-called Krall discrete orthogonal polynomials. A Krall discrete orthogonal family is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$, p_n of degree n, orthogonal with respect to a positive measure which, in addition, are also eigenfunctions of a higher order difference operator. A huge amount of families of Krall discrete orthogonal polynomials have been recently introduced by the author by means of certain Christoffel transform of the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see [10,11,14]). A Christoffel transform is a transformation which consists in multiplying a measure μ by a polynomial r. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E.B. Christoffel (see [7] and also [37]) studied it for the particular case $r(x) = x$.

In this paper we will concentrate on exceptional Charlier and Hermite polynomials (Meixner, Krawtchouk, Hahn, Laguerre and Jacobi families will be considered in the forthcoming papers).

The content of this paper is as follows. In Section 2, we include some preliminary results about symmetric operators, Christoffel transforms and finite sets of positive integers.