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Journal of Approximation Theory 182 (2014) 59–67

Journal of Approximation Theory

www.elsevier.com/locate/jat

Notes

A note on a Stone-Weierstrass type theorem for set-valued mappings

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Received 19 March 2013; received in revised form 20 January 2014; accepted 5 March 2014
Available online 18 March 2014

Communicated by Heinz Bauschke

Abstract

We establish a Stone-Weierstrass type theorem for set-valued mappings.

Keywords: Stone-Weierstrass theorem; Zorn lemma; Set-valued mapping; Property V

1. Introduction and preliminaries

Throughout this paper we shall assume, unless stated otherwise, that X is a locally compact Hausdorff space and $(E, \|\cdot\|)$ is a normed vector space over $\mathbb{K}$, where $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. We shall denote by $C(X; E)$ the vector space over $\mathbb{K}$ of all continuous functions from X into E and we denote by $C_b(X; \mathbb{K})$ the algebra of all bounded continuous $\mathbb{K}$-valued functions on X.

A continuous function f from X to E is said to *vanish at infinity* if for every $\varepsilon > 0$ the set $\{x \in X : \|f(x)\| \geq \varepsilon\}$ is compact. Let $C_0(X; E)$ be the vector space of all continuous functions from X into E vanishing at infinity and equipped with the supremum norm.

If Y is a nonempty subset of X and $f : X \to S$ is any function where S is a nonempty set, we denote by $f|_Y$ the function $y \in Y \mapsto f(y)$. If $\mathcal{F}$ is an arbitrary nonempty family of functions $f : X \to S$, we denote by $\mathcal{F}|_Y$ the set $\{f|_Y : f \in \mathcal{F}\}$.

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http://dx.doi.org/10.1016/j.jat.2014.03.008
0021-9045/

For $A \subset C(X; \mathbb{R})$, we define the following equivalence relation on $X : x \sim t \bmod A$ if and only if $g(x) = g(t)$ for all $g \in A$. For any $x \in X$, we denote by $[x]_A$ the equivalence class of x.

Let us denote by $D(X)$ the set $\{f \in C(X; \mathbb{R}) : 0 \le f \le 1\}$.

A subset A of $D(X)$ is said to have *property* V if it satisfies the following conditions:

(a) if $\phi \in A$, then $1 - \phi \in A$;
(b) if $\phi \in A$ and $\psi \in A$, then $\phi\psi \in A$.

The definition of property V was introduced by von Neumann [11].

If W is a nonempty subset of $C(X; E)$, a function $\phi \in D(X)$ is called a multiplier of W, if $\phi f + (1 - \phi)g \in W$ for every $f, g \in W$. We denote by $M(W)$ the set of multipliers of W. The notion of a multiplier of W is due to Feyel and De La Pradelle [3] and Chao-Lin [2].

Clearly, if $\phi \in M(W)$, then $1 - \phi \in M(W)$. Moreover the identity

$$\phi\psi f + (1 - \phi\psi)g = \psi[\phi f + (1 - \phi)g] + (1 - \psi)g$$

shows that $\phi\psi \in M(W)$, whenever $\phi, \psi \in M(W)$. Thus, $M(W)$ has property V.

A vector subspace A of $C(X; \mathbb{K})$ is called a subalgebra over $\mathbb{K}$ if $fg \in A$ whenever $f, g \in A$. It is self-adjoint if the complex-conjugate $\overline{f} = u - iv$ belongs to A, whenever $f = u + iv$ belongs to A.

Let A be a subalgebra of $C(X; \mathbb{K})$. A vector subspace $W \subset C(X; E)$ is called an A-module if $\phi g \in W$ for every $\phi \in A$ and $g \in W$.

A carrier is a set-valued mapping $\phi : X \to 2^E$ such that $\phi(x) \neq \emptyset$ for each $x \in X$.

The distance of a carrier $\phi : X \to 2^E$ from a function $g \in C_0(X; E)$ is defined by

$$d(\phi; g) := \sup_{x \in X} \sup_{y \in \phi(x)} \|y - g(x)\|$$

and the distance of ϕ from a subset $W \subset C_0(X; E)$ is

$$d(\phi; W) := \inf\{d(\phi; g) : g \in W\}.$$

If $v \in E$ and $s > 0$, we denote by $B(v, s)$ the set $\{u \in E : \|u - v\| < s\}$.

A carrier $\phi : X \to 2^E$ is said to be *upper semicontinuous with respect to* $W \subset C_0(X; E)$, if given $w \in W$ and $r > 0$, for each $x \in X$ such that $\phi(x) \subset B(w(x); r)$ and each $\varepsilon > 0$, there is a neighborhood U of x such that $\phi(y) \subset B(w(y); r + \varepsilon)$ for all $y \in U$.

A carrier $\phi : X \to 2^E$ is said to vanish at infinity with respect to $W \subset C_0(X; E)$, if for each $w \in W$ and $\varepsilon > 0$ the set

$$\{x \in X : \phi(x) \cap (E \setminus B(w(x); \varepsilon)) \neq \emptyset\}$$

is relatively compact.

We denote the closure of a subset T of a topological space by $\overline{T}$.

Prolla and Machado [8] proved the following “localization formula”.

Theorem 1.1. *Let $A \subset C_b(X; \mathbb{K})$ be a self-adjoint subalgebra and let $W \subset C_0(X; E)$ be an A-module. For any carrier $\phi : X \to 2^E$ which is upper semicontinuous and vanishes at infinity with respect to W, we have*

$$d(\phi; W) = \sup_{x \in X} d(\phi|_{[x]_A}; W|_{[x]_A}).$$

Soares [10] proved a version of Theorem 1.1 when E is a non-Archimedean normed space.

The Stone-Weierstrass theorem is a special case of Theorem 1.1.