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Full length article

The Bernstein inequality and the Schur inequality are equivalent☆

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Abstract

It is shown that the Bernstein Inequality and the Schur Inequality are equivalent in the sense that each can be obtained from the other one with the aid of brief elementary arguments.

Keywords: The Bernstein inequality; The Schur inequality; Marcel Riesz

1. Introduction

This paper consists of a theorem and its proof that comes straight from *The Book*. It is one those rare gems that lurked in front of our eyes for just over one hundred years now but it took my passionate and relentless pursuit of the true story of the Bernstein Inequality to become the first person to state the result explicitly although the proof itself, consisting of two independent components, has been around on the average of almost sixty years.

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[1] Although I, Paul N, wrote the paper, the referee's ingenious suggestion to rearrange a few lines resulted in a significantly better presentation giving a deep insight to the essence of the Schur Inequality, see (2.4).

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I decided to spoon-feed this theorem to the approximation theory community because I made an extensive survey within our ranks just to find out that not a single person was aware of it, not even the top experts on the Bernstein Inequality, although some of them knew parts of it. Specifically, those who read and remembered Pólya-Szegő knew one part but no one, with the exception of perhaps four persons, one them dead, two quite old, even older than I am, and one still active, knew of the other part.[2] I wish I could tell the full story here but then there would be no readership left for my upcoming book [11].

2. The result and the proof

In what follows, $n \in \mathbb{N}$ is fixed. Let $\mathbb{P}_n$ and $\mathbb{T}_n$ be set of algebraic and trigonometric polynomials of degree at most n with complex coefficients, respectively. Let $\|\cdot\|_S$ denote the supremum norm on the set S.

Every approximator[3] knows the Bernstein Inequality[4] that says that

$$\left\|T'\right\|_{\mathbb{R}} \leqslant n \,\|T\|_{\mathbb{R}} \qquad \forall T \in \mathbb{T}_n, \tag{2.1}$$

and almost every approximator knows that[5]

$$\|P\|_{[-1,1]} \leqslant n \left\|\sqrt{1-(\cdot)^2}\, P(\cdot)\right\|_{[-1,1]} \qquad \forall P \in \mathbb{P}_{n-1}, \tag{2.2}$$

although not all of them know that it's usually called Schur's Inequality or Schur's Lemma with a standard reference to Schur's 1919 paper [17] even if Schur never in his life stated (2.2) explicitly.

Clearly, by a change of variables $x = \cos\varphi$, (2.2) is equivalent to

$$\left\|\frac{T(\cdot)}{\sin(\cdot)}\right\|_{\mathbb{R}} \leqslant n \,\|T\|_{\mathbb{R}} \qquad \forall \text{ odd } T \in \mathbb{T}_n, \tag{2.3}$$

and this is how M. Riesz stated and proved (2.2) in 1914, see [15, Sätze II & III, pp. 358–359].

Perhaps the simplest way to state (2.2) is

$$|P(1)| \leqslant n \left\|\sqrt{1-(\cdot)^2}\, P(\cdot)\right\|_{[-1,1]} \qquad \forall P \in \mathbb{P}_{n-1}. \tag{2.4}$$

Amazingly, (2.4) implies (2.2) immediately by applying the former to $P(u\omega)$ where ω is the variable and $u \in [-1,1]$ is "just a parameter" so that

$$|P(u)| \leqslant n \max_{\omega\in[-1,1]} \left(\sqrt{1-\omega^2}\; |P(u\omega)|\right),$$

and, therefore, since $\sqrt{1-\omega^2} \leqslant \sqrt{1-(u\omega)^2}$ for $u \in [-1,1]$,

$$|P(u)| \leqslant n \max_{\omega\in[-1,1]} \left(\sqrt{1-(u\omega)^2}\; |P(u\omega)|\right),$$

[2] These are the ones in whose papers I read it. I use the word "perhaps" because I failed to obtain an answer from the live ones about the origins of their proof except that they all explicitly stated to me that it was not their own original idea.

[3] approximator $\equiv$ approximation theorist.

[4] Bernstein neither formulated nor proved (2.1) first. Even his proof of a weaker statement was partially defective that took him some 40 years of tinkering to fix. If credit must be given then it should go to M. Riesz; for the juicy details see [11].

[5] I know this $(\cdot)$ notation is hideously ugly but I just can't force myself to write expressions such as $\left\|\sqrt{1-x^2}\, P(x)\right\|_{[-1,1]}$ and the max-notation is too old fashioned for my taste.