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Journal of Approximation Theory 184 (2014) 1–27

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Critically large entropy of absolutely convex hulls in Banach spaces of type p

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Received 31 May 2013; received in revised form 27 January 2014; accepted 1 May 2014
Available online 10 May 2014

Communicated by Ding-Xuan Zhou

Abstract

Given a precompact subset A of a Banach space of type p, we investigate and quantify in which way the rate of decay of the entropy numbers of A affects the rate of decay of the entropy numbers of the real absolutely convex hull of A. This problem has been studied by various researchers in different settings during the last 25 years. We get new and best possible results both in the case of logarithmic decay of entropy numbers of A and in the case that the entropy numbers of A belong to some Lorentz sequence space making a major step to completing the picture.

MSC: 41A46; 46B20; 52A07

Keywords: Entropy number; Covering number; Metric entropy; Convex hull; Type p space

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http://dx.doi.org/10.1016/j.jat.2014.05.001
0021-9045/

1. Introduction and results

Given a precompact subset A of a Banach space X it is common knowledge that also the real absolutely convex hull $\mathrm{aco}(A)$ of A, given by

$$\mathrm{aco}(A) = \left\{ \sum_{i=1}^{n} \lambda_i \, a_i \mid n \in \mathbb{N}, a_i \in A, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\},$$

is precompact, i.e.

$$A \text{ precompact} \quad \text{implies} \quad \mathrm{aco}(A) \text{ precompact.} \tag{1.1}$$

For many applications such as the eigenvalue behavior of linear operators or the path properties of Gaussian stochastic processes, quantifying the degree of precompactness of $\mathrm{aco}(A)$ in dependence of the degree of precompactness of A is most fruitful. To measure precompactness we refer to the notion of entropy numbers of a set. Throughout this paper, $B_M(c, r)$ denotes the closed ball in the metric space M with center $c \in M$ and radius $r \geq 0$.

Definition. Let (X, d) be a metric space and $A \subset X$ a precompact set. For a natural number n, the *nth entropy number of* A is defined by

$$\varepsilon_n(A; X) := \inf \left\{ \varepsilon \geq 0 \mid \exists x_1, \ldots, x_n \in X : A \subset \bigcup_{i=1}^{n} B_X(x_i, \varepsilon) \right\}.$$

The *n-th dyadic entropy number* of A is given by

$$e_n(A; X) := \varepsilon_{2^{n-1}}(A; X), \quad n = 1, 2, 3, \ldots.$$

We write $\varepsilon_n(A)$ instead of $\varepsilon_n(A; X)$ for short if there is no danger of confusion.

Alternatively, one can employ covering numbers. This concept dates back to 1932, see Pontrjagin and Schnirelmann [30, p. 156]. Entropy numbers and covering numbers are, in a sense, inverse quantities. For more information we refer to [12,26,28,13,17].

Definition. Let (X, d) be a metric space and $A \subset X$ a precompact set. For $\varepsilon > 0$, the *ε-covering number of* A is defined by

$$N(A, \varepsilon) := \min \left\{ n \in \mathbb{N} \mid \exists x_1, \ldots, x_n \in X : A \subset \bigcup_{i=1}^{n} B_X(x_i, \varepsilon) \right\}.$$

Note that A is precompact if and only if $\lim_{n\to\infty} \varepsilon_n(A) = 0$. Hence, implication (1.1) can be reformulated in the language of entropy numbers as

$$\lim_{n\to\infty} \varepsilon_n(A) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \varepsilon_n(\mathrm{aco}(A)) = 0.$$

The rates of decay of $\varepsilon_n(A)$ and $\varepsilon_n(\mathrm{aco}(A))$ as $n \to \infty$ can be considered to quantify the degree of precompactness of the set A and its absolutely convex hull $\mathrm{aco}(A)$. The way in which the former affects the latter will be subject of our studies.

The first who formulated and treated this problem in a general way was Dudley [16] in the Hilbert space setting motivated by applications to empirical processes. However, it should be noted that the study of operators acting from an l_1-space into a Banach space implicitly contained