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No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms

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Abstract

We show that none of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \leq p \neq q < \infty$ have a greedy basis. This solves a problem raised by Dilworth, Freeman, Odell and Schlumprecht. Similarly, the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \leq p < \infty$, and $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \leq q < \infty$, do not have greedy bases. It follows from that and known results that a class of Besov spaces on $\mathbb{R}^n$ lack greedy bases as well.
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1. Introduction

Given a (say, real) Banach space X with a Schauder basis $\{x_i\}$, an $x \in X$ and an $n \in \mathbb{N}$ it is useful to determine the best n-term approximation to x with respect to the given basis. I.e., to find a set $A \subset \mathbb{N}$ with n elements and coefficients $\{a_i\}_{i \in A}$ such that

$$\left\| x - \sum_{i \in A} a_i x_i \right\| = \inf \left\{ \left\| x - \sum_{i \in B} b_i x_i \right\| ; |B| = n, b_i \in \mathbb{R} \right\}$$

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or, given a $C < \infty$, at least to find such an $A \subset \mathbb{N}$ and coefficients $\{a_i\}_{i \in A}$ with

$$\left\| x - \sum_{i \in A} a_i x_i \right\| \le C \inf \left\{ \left\| x - \sum_{i \in B} b_i x_i \right\| ; |B| = n, b_i \in \mathbb{R} \right\}.$$

This problem attracted quite an attention in modern Approximation Theory. Of course one would also like to have a simple algorithm to find such a set $\{a_i\}_{i \in A}$. It would be nice if we could take $\{a_i\}_{i \in A}$ to be just the set of the n largest, in absolute value, coefficients in the expansion of x with respect to the basis $\{x_i\}$. Or, if this set is not unique, any such set. The basis $\{x_i\}$ is called *Greedy* if for some C this procedure works; i.e., for all $x = \sum_{i=1}^{\infty} a_i x_i$, all $n \in \mathbb{N}$ and all $A \subset \mathbb{N}$, $|A| = n$, satisfying $\min\{|a_i|; i \in A\} \ge \max\{|a_i|; i \notin A\}$,

$$\left\| x - \sum_{i \in A} a_i x_i \right\| \le C \inf \left\{ \left\| x - \sum_{i \in B} b_i x_i \right\| ; |B| = n, b_i \in \mathbb{R} \right\}.$$

Konyagin and Temlyakov [4] provided a simple criterion to determine whether a basis is greedy: $\{x_i\}$ is greedy if and only if it is *unconditional* and *democratic*.

Recall that $\{x_i\}$ is said to be unconditional provided, for some $C < \infty$, all eventually zero coefficients $\{a_i\}$ and all sequences of signs $\{\varepsilon_i\}$,

$$\left\| \sum \varepsilon_i a_i x_i \right\| \le C \left\| \sum a_i x_i \right\|.$$

$\{x_i\}$ is said to be democratic provided for some $C < \infty$ and all finite $A, B \subset \mathbb{N}$ with $|A| = |B|$,

$$\left\| \sum_{i \in A} x_i \right\| \le C \left\| \sum_{i \in B} x_i \right\|.$$

We refer to [2] for a survey of what is known about space that have or do not have greedy bases. In [2] Dilworth, Freeman, Odell and Schlumprecht determined which of the spaces $X = (\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \le p \ne q \le \infty$ (with c_0 replacing ℓ_∞ in case $q = \infty$) have a greedy basis. It turns out that this happens exactly when X is reflexive. They also raise the question of whether $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 < p \ne q < \infty$ have greedy bases. Here we show that these spaces (as well as their non-reflexive counterparts) do not have greedy bases. By the Konyagin–Temlyakov characterization it is enough to prove that each normalized unconditional basis of $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \le p \ne q \le \infty$ (with c_0 replacing ℓ_∞ in case p or q are ∞) has two subsequences, one equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$) and one to the unit vector basis of ℓ_q (c_0 if $q = \infty$).

Theorem 1. *Each normalized unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \le p \ne q < \infty$ has a subsequence equivalent to the unit vector basis of ℓ_p and another one equivalent to the unit vector basis of ℓ_q. Similarly, each normalized unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \le p < \infty$ (resp. $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \le q < \infty$) has a subsequence equivalent to the unit vector basis of ℓ_p (resp. c_0) and another one equivalent to the unit vector basis of c_o (resp. ℓ_q). Consequently, none of these spaces have a greedy basis.*

For $1 \le p, q < \infty$ the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$ are isomorphic to certain Besov spaces on $\mathbb{R}^n$. We refer to [8] for the definition of the Besov spaces $B_p^{s,q}$ and for the fact that they are isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$. See in particular [8, Section 6.10, Proposition 7] (and also [8, Section 2.9, Proposition 4]). We thank P. Wojtaszczyk for this reference.

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