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Zak transforms and Gabor frames of totally positive functions and exponential B-splines

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Abstract

We study totally positive (TP) functions of finite type and exponential B-splines as window functions for Gabor frames. We establish the connection of the Zak transform of these two classes of functions and prove that the Zak transforms have only one zero in their fundamental domain of quasi-periodicity. Our proof is based on the variation-diminishing property of shifts of exponential B-splines. For the exponential B-spline B_m of order m, we determine a set of lattice parameters $\alpha, \beta > 0$ such that the Gabor family $\mathcal{G}(B_m, \alpha, \beta)$ of time–frequency shifts $e^{2\pi i l \beta} B_m(\cdot - k\alpha)$, $k, l \in \mathbb{Z}$, is a frame for $L^2(\mathbb{R})$. By the connection of its Zak transform to the Zak transform of TP functions of finite type, our result provides an alternative proof that TP functions of finite type provide Gabor frames for all lattice parameters with $\alpha\beta < 1$. For even two-sided exponentials $g(x) = \frac{\lambda}{2} e^{-\lambda|x|}$ we find lower frame-bounds A, which show the asymptotically linear decay $A \sim (1 - \alpha\beta)$ as the density $\alpha\beta$ of the time–frequency lattice tends to the critical density $\alpha\beta = 1$.

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Keywords: Gabor frame; Total positivity; Exponential B-spline; Zak transform

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0. Introduction

The Gabor transform provides an important tool for the analysis of a given signal $f : \mathbb{R} \to \mathbb{C}$ in time and frequency. It is a fundamental tool for time–frequency analysis and serves various purposes, such as signal denoising or compression. A window function $g \in L^2(\mathbb{R})$ has time–frequency shifts

$$M_\xi T_y g(x) = e^{2\pi i \xi x} g(x-y), \quad \xi, y \in \mathbb{R}.$$

The Gabor transform of a square-integrable signal f is defined as

$$\mathcal{S}_g f(k\alpha, l\beta) = \langle f, M_{l\beta} T_{k\alpha} g\rangle, \quad k, l \in \mathbb{Z},$$

where the parameters $(k\alpha, l\beta) \in \alpha\mathbb{Z} \times \beta\mathbb{Z}$ of the time–frequency shifts of g form a lattice in $\mathbb{R}^2$, with lattice parameters $\alpha, \beta > 0$. The family

$$\mathcal{G}(g, \alpha, \beta) := \{M_{l\beta} T_{k\alpha}\, g \mid k, l \in \mathbb{Z}\}$$

is called a Gabor family. An important problem in Gabor analysis is to determine, for a given window function $g \in L^2(\mathbb{R})$ and lattice parameters $\alpha, \beta > 0$, if the Gabor family $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$. This means that there exist constants $A, B > 0$, which depend on g, α, β, such that for every $f \in L^2(\mathbb{R})$ we have

$$A\|f\|^2 \le \sum_{k,l\in\mathbb{Z}} |\langle f, M_{l\beta} T_{k\alpha} g\rangle|^2 \le B\|f\|^2. \tag{1}$$

Moreover, it is interesting for the selection of sampling and modulation rates in signal analysis to know the whole *frameset*

$$\mathcal{F}_g := \{(\alpha, \beta) \in \mathbb{R}_+^2 \mid \mathcal{G}(g, \alpha, \beta) \text{ is a frame}\}$$

of a given function $g \in L^2(\mathbb{R})$. Until 2013, the framesets of only a few functions, like the one- and two-sided exponential [18,19], the Gaussian [25,35], the hyperbolic secant [21] and the characteristic function $\chi_{[0,1)}$ [7] were known. Then it was proved in [15, Theorem 1] that the frameset of a large class of functions, the *totally positive functions of finite type* $m \ge 2$, is given by

$$\mathcal{F}_g = \mathbb{H} := \{(\alpha, \beta) \in \mathbb{R}_+^2 \mid 0 < \alpha\beta < 1\}.$$

This result complements the known fact, that for continuous window functions of the *Wiener space*,

$$W(\mathbb{R}) := \left\{ g \in L^\infty(\mathbb{R}) \mid \|g\|_W = \sum_{n\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in[0,1]} |g(x+n)| < \infty \right\},$$

the frameset is a subset of $\mathbb{H}$.

In order to describe Gabor families of a window function g, a very helpful tool is the Zak transform

$$\mathrm{Z}_\alpha g(x, \omega) := \sum_{k\in\mathbb{Z}} g(x - k\alpha) e^{2\pi i k\alpha\omega}, \quad (x, \omega) \in \mathbb{R}^2.$$

In Approximation Theory, the Zak transform was used by Schoenberg [31] in connection with cardinal spline interpolation. For a polynomial B-spline B_m of degree $m-1$, Schoenberg called