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Total positivity of a Cauchy kernel

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Abstract

We study the total positivity of the kernel $1/(x^2 + 2\cos(\pi\alpha)xy + y^2)$ over $\mathbb{R}^+ \times \mathbb{R}^+$. The case of infinite order is characterized by an application of Schoenberg's theorem. We give necessary conditions for the case of finite order with the help of Chebyshev polynomials of the second kind. Sufficient conditions for the case of finite order are obtained thanks to the Izergin–Korepin determinant and its expression in terms of alternating sign matrices. A conjecture on the generating function of alternating sign matrices with a fixed number of negative entries arises in a natural way from our study. As a by-product, we give a partial answer to a question of Karlin on positive stable semigroups.
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1. Introduction

Let I be a real interval and K a real kernel defined on $I \times I$. The kernel K is called totally positive of order n (TP_n) if

$$\det\left[K(x_i, y_j)\right]_{1\le i,j\le m} \ge 0$$

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for every $m \in \{1, \ldots, n\}$, $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$. If these inequalities hold for all n one says that K is TP_∞. The kernel K is called sign-regular of order n (SR_n) if there exists $\{\varepsilon_m\}_{1 \le m \le n} \in \{-1, 1\}$ such that

$$\varepsilon_m \det \left[K(x_i, y_j) \right]_{1 \le i, j \le m} \ge 0$$

for every $m \in \{1, \ldots, n\}$, $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$. In the case $\varepsilon_m = (-1)^{m(m-1)/2}$ for all $m \in \{1, \ldots, n\}$, the kernel K is called reverse-regular of order n (RR_n). If these inequalities hold for all n one says that K is SR_∞ resp. RR_∞. The above properties are said to be strict, with corresponding notations STP, SSR, and SRR, when all involved inequalities are strict. We refer to [13] for the classic account on this field and its various connections with analysis, especially Descartes' rule of signs. Let us also mention the survey paper [14] for a statistical point of view on total positivity, and the recent monograph [21] for a linear algebraic point of view with updated and historical references.

A function $f : \mathbb{R} \to \mathbb{R}^+$ is called a Pólya frequency function of order $n \le \infty$ (PF_n) if the kernel $K(x, y) = f(x - y)$ is TP_n on $\mathbb{R} \times \mathbb{R}$. When this kernel is STP_n, we will use the notation $f \in \text{SPF}_n$. Probability densities belonging to the class PF_∞ have been characterized by Schoenberg – see e.g. Theorem 7.3.2 (i) p. 345 in [13] – through the meromorphic extension of their Laplace transform, whose reciprocal is an entire function having the following Hadamard factorization:

$$\frac{1}{\mathbb{E}[e^{sX}]} = e^{-\mu s^2 + \delta s} \prod_{n=0}^{\infty} (1 + a_n s) e^{-a_n s} \tag{1.1}$$

with X the associated random variable, $\mu \ge 0$, $\delta \in \mathbb{R}$, and $\sum a_n^2 < \infty$. The classical example is the Gaussian density. Pólya frequency functions of order 2 are easily characterized by the log-concavity on their support—see Theorems 4.1.8 and 4.1.9 in [13]. But except for the cases $n = 2$ or $n = \infty$ there is no handy criterion for testing the PF_n character of a given function resp. the TP_n character of a given kernel, and such questions might turn out to be difficult. In this paper we consider the simple kernel

$$K_\alpha(x, y) = \frac{1}{x^2 + 2\cos(\pi\alpha) xy + y^2}$$

over $(0, +\infty) \times (0, +\infty)$, with $\alpha \in [0, 1)$. Because of its proximity to the standard Cauchy density we may and will call K_α a Cauchy kernel, although no confusion should be made with the traditional Cauchy kernel from complex analysis. Up to prefactors, the kernel K_α is the multiplicative convolution kernel associated with a Cauchy distribution or the additive convolution kernel associated with a generalized logistic distribution. The monograph [13] displays total positivity properties of many density functions, but it seems that in this respect the two above very classical distributions have escaped investigation so far. As will be seen below, the kernel K_α has also close connections to some aspects of the statistical mechanical six-vertex model with domain-wall boundary conditions. Our main result is the following.

Theorem. (a) *One has*

$$K_\alpha \in \text{STP}_\infty \iff K_\alpha \in \text{SR}_\infty \iff \alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots, 0\}.$$

(b) *For every positive integer n one has*

$$K_\alpha \in \text{TP}_n \iff K_\alpha \in \text{SR}_n \implies \alpha \in \{1/2, 1/3, \ldots, 1/n\} \quad \text{or } \alpha \le 1/n.$$

(c) *One has* $K_\alpha \in \text{STP}_n$ *if* $n \in \{1, \ldots, 6\}$ *and* $\alpha \le 1/n$ *or* $n \ge 7$ *and* $\alpha \le 1/(n^2 - n - 6)$.

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