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Doubly universal Taylor series

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Abstract

For a holomorphic function f in the unit disk, $S_n(f)$ denotes the n-th partial sum of the Taylor development of f with center at 0. We show that given a strictly increasing sequence of positive integers (λ_n), there exists a holomorphic function f on the unit disk such that the pairs of partial sums $\{(S_n(f), S_{\lambda_n}(f)) : n = 1, 2, \dots\}$ approximate all plausibly approximable functions uniformly on suitable compact subsets K of the complex plane if and only if $\limsup_n \frac{\lambda_n}{n} = +\infty$. This provides a new strong notion of universality for Taylor series.

Keywords: Universal Taylor series; Double universality; Walsh–Bernstein theorem; Degree of approximation

1. Introduction

We introduce first some standard notation and terminology. As usual D denotes the open unit disk of the complex plane $\mathbb{C}$ centered at 0. For an open set U in $\mathbb{C}$ the symbol $H(U)$ stands for the set of holomorphic functions on U and $H(U)$ becomes a complete topological vector space under the topology inherited by the uniform convergence on all the compact subsets of U. Let X be a complete metric space. A subset of X is called G_δ if it can be written as a countable intersection of open sets in X and a subset of X is called residual if it contains a G_δ and dense set in X. Also, for K a compact set of $\mathbb{C}$ we denote by $A(K)$ the set of all functions which are holomorphic in the interior K^o of K and continuous on K. For a given complex number z and a

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positive number $r > 0$ the symbols $D(z,r)$, $\overline{D(z,r)}$ denote the open and closed disks centered at z with radius r respectively. Finally, $\mathbb{N}$, $\mathbb{Q}$ denote the sets of positive integers and rational numbers respectively.

The notion of universality of Taylor series has a long history and as far as we know the first result in this direction is due to Fekete, see [25]. In this work we concentrate on a universal property of Taylor series which was firstly noticed and studied by Nestoridis in [24]; for a similar, but weaker, kind of universal property of Taylor series see [8,17,18]. We recall the main definition from [24]. A function $f \in H(D)$ is called a *universal Taylor series*, if the sequence of its partial sums, $(S_n(f))$ (with center 0), has the following property: for every compact set $K \subset \mathbb{C}$ with K^c connected and $K \cap D = \emptyset$ and for every function $g \in A(K)$ there exists a sequence of positive integers (m_n) such that $\|S_{m_n}(f) - g\|_K \to 0$, where $\|h\|_K := \sup_{z\in K} |h(z)|$ for every function $h : K \to \mathbb{C}$ continuous on K. The set of universal Taylor series is denoted by $U(D, 0)$. Roughly speaking, the partial sums of a universal Taylor series have the "wildest" possible behavior. The author's main result in [24] is to establish the existence of universal Taylor series via a Baire category argument. Properties of universal Taylor series as well as several extensions and refinements of the above definition can be found, for instance, in [3] and the references therein. We introduce the following new form of universality for Taylor series.

Definition 1.1. Let (λ_n) be a strictly increasing sequence of positive integers. A function $f \in H(D)$ belongs to the class $U(D, (\lambda_n), 0)$ if for every compact set $K \subset \mathbb{C}$ with K^c connected and $K \cap D = \emptyset$ and for every pair of functions $(g_1, g_2) \in A(K) \times A(K)$, there exists a subsequence of positive integers (μ_n) such that

$$\|S_{\mu_n}(f) - g_1\|_K \to 0 \quad \text{and} \quad \|S_{\lambda_{\mu_n}}(f) - g_2\|_K \to 0.$$

Such a function f will be called *doubly universal Taylor series with respect to the sequences* (n), (λ_n).

The term "doubly universal functions" or more precisely "multiply universal functions" also appears in [19], but with a different meaning than in our case. We would like to stress that Definition 1.1 is inspired by the following two central notions in dynamical systems and ergodic theory: (i) topological multiple recurrence and (ii) disjointness. The first one in its simplest form goes back to Poincaré, while the second one, which is more recent, is due to Furstenberg, [9]. To explain briefly, consider a dynamical system (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. The famous theorem of Poincaré asserts the existence of recurrent points for T, i.e. there exists $x \in X$ such that $T^{k_n}x \to x$ along some strictly increasing sequence (k_n) of positive integers, where $(T^n x)$ denotes the sequence of iterates of x under T. Later, Furstenberg and Weiss proved a most general version of this phenomenon which is now known as Furstenberg–Weiss topological multiple recurrence theorem [10]: if X is a compact metric space and $T_1, T_2, \ldots, T_m$ are commuting homeomorphisms of X then some $x \in X$ is multiply recurrent, that is, there exists a sequence of positive integers (k_n) with $k_n \to +\infty$ such that $T_j^{k_n}x \to x$ for every $j = 1, \ldots, m$. In particular, if $m = 2$, $T_1 = T$ and $T_2 = T^2$ then there exist $x \in X$ and a sequence of positive integers (k_n) with $k_n \to +\infty$ such that $T^{k_n}x \to x$ and $T^{2k_n}x \to x$. This means that the pair (x, x) is a limit point of the sequence $((T^n x, T^{2n}x))$. The notion of disjointness, see [9] for the precise definition, is a much stronger property which somehow reflects the density of diagonal orbits. For instance, under some natural assumptions on the dynamical systems (X, T), (X, T^2) the notion of disjointness implies the existence of points $x \in X$ such that the set $\{(T^n x, T^{2n}x) : n = 0, 1, 2, \ldots\}$ is dense in $X \times X$, see [9]. It seems