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New moduli of smoothness on the unit ball, applications and computability

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Abstract

In a previous article I introduced new moduli of smoothness on the unit ball and derived their main properties. The usefulness of these moduli is demonstrated here by some applications. Ul'yanov-type inequalities are derived via new Nikolski-type inequalities. Sharp Jackson inequalities are given for some Banach spaces whose unit ball satisfies a certain convexity condition. Connections with appropriate Besov spaces are discussed. Computations of the moduli for some functions are exhibited. This will illustrate the situations and the optimality of the conditions in some theorems as well as demonstrate the computability of the moduli.
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1. Introduction

For the unit ball $B^d \subset R^d$, $H_n(\lambda, d)$ is the space of all polynomials of degree n which are orthogonal to all polynomials of degree $n-1$ with respect to the weight $(1-\|\boldsymbol{x}\|^2)^{\lambda-\frac{1}{2}}$, $\lambda > -\frac{1}{2}$. The semigroup $T_\alpha(t)$ is given by

$$T_\alpha(t)f \equiv T_\alpha(\lambda, d, t)f \sim \sum_{n=0}^{\infty} \exp\left(-tn^\alpha (n+2\lambda-1)^\alpha\right) P_n f \tag{1.1}$$

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where $P_n f$ is the L_2 projection of f on $H_n(\lambda, d)$. The semigroup $\widetilde{T}_\alpha(t)f = \widetilde{T}_\alpha(\lambda, d, t)f$ is given by replacing $\exp\bigl(-tn^\alpha(n+2\lambda-1)^\alpha\bigr)$ by e^{-tn^α}. The moduli of smoothness $\omega^\beta(f,t)_B$ and $\widetilde{\omega}^\beta(f,t)_B$ are given by

$$\omega^{2\beta}(f,t)_B = \|T_\beta(t^{2\beta})f - f\|_B, \qquad \widetilde{\omega}^\beta(f,t)_B = \|\widetilde{T}_\beta(t^\beta)f - f\|_B \tag{1.2}$$

for the Banach space B. By $\widehat{\omega}^\beta(f,t)_B$ we denote either $\omega^\beta(f,t)_B$ or $\widetilde{\omega}^\beta(f,t)_B$. Many results were applicable only to the weight with $\lambda \ge 0$. As too many concepts, definitions and results were given in [11] and as this paper is in fact a continuation of that paper, we will just cite them when used with formula number or theorem number, etc. I note that while the concepts in (1.2) are close, they are not necessarily equivalent.

In Section 2 we will give the Nikolski inequality and show that the powers achieved are best possible. In Section 3 we derive Ul'yanov type inequalities from the results of Section 2. In Section 4 we obtain a sharp Jackson inequality for those Banach spaces whose unit ball has a "modulus of convexity of power type s", which include $L_{p,\lambda}(B^d)$ $1 < p < \infty$ where $s = \max(p, 2)$. In Section 5 connections with appropriate Besov spaces are discussed. Finally, computation of the moduli for various functions which also illustrate some of the previous results will be given in Section 6. As explained in the introduction of [11], the moduli introduced there are different and have some advantages over the moduli in [7]. In Section 7 some additional results and questions are discussed. In this paper the functions and polynomials are assumed to be real valued. However, most of the results are valid for complex valued functions and polynomials with little or no extra work.

2. Nikolski inequality on the unit ball

In this section we will prove a Nikolski-type estimate for polynomials on B^d, that is, we will determine $\sigma = \sigma(d,\lambda)$ for which $\|\varphi\|_{L_{q,\lambda}(B^d)} \le Cn^{\sigma\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{L_{p,\lambda}(B^d)}$ when $\varphi \in \operatorname{span}\left(\cup_{k=0}^{n-1} H_k\right)$, i.e. φ is a polynomial of total degree $< n$, $\mu(\boldsymbol{x}) = (1-\|\boldsymbol{x}\|^2)^{\lambda-\frac{1}{2}}$, $\lambda > -\frac{1}{2}$ and $0 < p < q \le \infty$. For the unit sphere a Nikolski-type inequality was proved by Kamzalov (see [17]) and a shorter proof was given in [1, Lemma 7.4]. For T_d, $\mu(\boldsymbol{x}) = 1$, a Nikolski type inequality was given in [14, Lemma 8.2].

In a recent paper F. Dai and H. Wang (see [6, Remark 4.4]) show that as a corollary of their cubature results, the Nikolski inequality for B^d with w_λ where $\lambda \ge 0$ and T_d with $w_{\boldsymbol{\gamma}}$ where $\gamma_i \ge -\frac{1}{2}$ can be deduced. The cases $-\frac{1}{2} < \lambda < 0$ for w_λ and $-1 < \gamma_i < -\frac{1}{2}$ for $w_{\boldsymbol{\gamma}}$ were not considered in [6]. Moreover, the Nikolski inequality was not shown in [6] to be optimal.

The determination of the Nikolski inequality will yield Ul'yanov-type inequalities for $\omega^\beta(f,t)_B$, $\widetilde{\omega}^\beta(f,t)_B$ and $E_n(f)_B$ for $B = L_{p,\lambda}(B^d)$. We hope that other estimates given in this section will be used in the future.

An orthonormal basis of $H_n(\lambda, d)$, $d \ge 2$ is given by (see [16, p. 39, Proposition 2.3.1])

$$P_{n,j,\ell}^{\lambda,d}(\boldsymbol{x}) = \left(2^{n-2j+\frac{d}{2}+1+\lambda}\right)^{1/2} p_j^{(\lambda-\frac{1}{2},n-2j+\frac{d-2}{2})}(2\|\boldsymbol{x}\|^2-1)S_{n-2j,\ell}(\boldsymbol{x}) \tag{2.1}$$

where $0 \le 2j \le n$, $1 \le \ell \le d_{n-2j} \equiv d_{n-2j}(S^{d-1})$, $d_k = d_k(S^{d-1}) = \frac{d+2k-2}{k}\binom{d+k-3}{k-1} \approx k^{d-2}$ for $k > 0$, $d_0 = 1$, $p_j^{(\alpha,\beta)}(t)$ are the normalized Jacobi polynomials of degree j, $S_{k,\ell}(\boldsymbol{x}) \equiv \|\boldsymbol{x}\|^k Y_{k,\ell}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)$ and $\{Y_{k,\ell}(\boldsymbol{y})\}_{\ell=1}^{d_k}$ is an orthonormal basis of the spherical harmonic polynomials

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