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Journal of Approximation Theory 179 (2014) 33–62

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Widths of embeddings of 2-microlocal Besov spaces

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Received 10 September 2012; received in revised form 10 October 2013; accepted 10 November 2013

Available online 8 December 2013

Communicated by Bin Han

Abstract

We consider the asymptotic behavior of the approximation, Gelfand and Kolmogorov numbers of compact embeddings between 2-microlocal Besov spaces with weights defined in terms of the distance to a d-set $U \subset \mathbb{R}^n$. The sharp estimates are shown in most cases, where the quasi-Banach setting is included.

Keywords: Approximation numbers; Gelfand numbers; Kolmogorov numbers; Compact embeddings; 2-microlocal Besov spaces

1. Introduction

In this paper we investigate the embeddings between 2-microlocal Besov spaces with a special type of weights from the standpoint of certain approximation quantities. More precisely, we are interested in asymptotic behavior of the approximation, Gelfand and Kolmogorov numbers. This problem has recently been suggested only for entropy numbers by Leopold and Skrzypczak [20]. First, we recall some definitions.

Let φ be a nonnegative function from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of infinitely differentiable and rapidly decreasing functions with

$$\varphi(x) = 1 \quad \text{for } |x| \le 1 \text{ and supp } \varphi \subset \{x : |x| \le 2\}. \tag{1.1}$$

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0021-9045/$ - see front matter
http://dx.doi.org/10.1016/j.jat.2013.11.005

We set $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the smooth dyadic resolution $\{\varphi_j\}_{j\in\mathbb{N}_0}$ of unity, i.e., $\sum_{j=0}^{\infty}\varphi_j(x) = 1,\ x \in \mathbb{R}^n$. The dual space of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions which we denote by $\mathcal{S}'(\mathbb{R}^n)$. Then

$$f = \varphi_0(\mathcal{D})f + \sum_{j=1}^{\infty}\varphi_j(\mathcal{D})f, \quad f \in \mathcal{S}'(\mathbb{R}^n),$$

where the symbol $\mathcal{D}$ is just ancillary to the definition as below,

$$\varphi_j(\mathcal{D})f(x) := (2\pi)^{-n}\iint e^{i\xi(x-y)}\varphi_j(\xi)f(y)dyd\xi.$$

For a bounded subset $U \subset \mathbb{R}^n$, we denote $\operatorname{dist}(x, U) = \inf_{y\in U}|x - y|$, and we define for $\sigma \in \mathbb{R}$ the 2-microlocal weights by

$$w_j(x) := (1 + 2^j \operatorname{dist}(x, U))^{\sigma}, \quad j \in \mathbb{N}_0,\ x \in \mathbb{R}^n. \tag{1.2}$$

This type of weight sequences is just a typical example for admissible weight sequences. We refer to [13,15] for detailed discussions of a large class of admissible weight sequences. The case of single weights seems more familiar to us, cf. [7,9,19,33–35].

Given $0 < p, q \le \infty$ and $s, \sigma \in \mathbb{R}$, we define the 2-microlocal spaces $B^{s,\sigma}_{p,q}(\mathbb{R}^n, U)$ by

$$B^{s,\sigma}_{p,q}(\mathbb{R}^n, U) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid B^{s,\sigma}_{p,q}(\mathbb{R}^n, U)\| < \infty\right\},$$

where

$$\|f \mid B^{s,\sigma}_{p,q}(\mathbb{R}^n, U)\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\|w_j\varphi_j(D)f \mid L_p(\mathbb{R}^n)\|^q\right)^{1/q}. \tag{1.3}$$

There is an analogous definition for the 2-microlocal Triebel–Lizorkin spaces $F^{s,\sigma}_{p,q}(\mathbb{R}^n, U)$, cf. [14]. Moritoh and Yamada introduced in [23] the spaces $B^{s,\sigma}_{p,q}(\mathbb{R}^n, U)$ of homogeneous type in case when $U \subset \mathbb{R}^n$ is open.

2-Microlocal Besov spaces $B^{s,mloc}_{p,q}(\mathbb{R}^n, w)$ with more general admissible weights were introduced by Kempka [13,16], and generalized the 2-microlocal spaces $C^{s,\sigma}_{x_0}(\mathbb{R}^n)$ introduced by Bony [1] and Jaffard [11] in two directions. Indeed, 2-microlocal spaces are a useful tool to measure the local regularity of functions and to study non-linear hyperbolic equations. We refer to [13–15,22] for systematic discussions of this concept, its history and further references.

Following Leopold and Skrzypczak [20], we concentrate on the embeddings,

$$B^{s_1,\sigma_1}_{p_1,q_1}(\mathbb{R}^n, U) \hookrightarrow B^{s_2,\sigma_2}_{p_2,q_2}(\mathbb{R}^n, U), \tag{1.4}$$

where U is a d-set (the precise definition of d-sets will be given in Section 2).

Our main intention in this paper is to find the optimal asymptotic order of the approximation, Gelfand and Kolmogorov numbers of the embeddings (1.4). Our approach is essentially a combination of [20,27] with its corrigendum [28]. In particular, Leopold and Skrzypczak [20] gave a necessary and sufficient condition on the parameters and weights of the 2-microlocal Besov spaces which guarantees compactness of the embeddings (1.4), and determined the entropy estimates for such embeddings. Moreover, our main tools are the use of operator ideals, see [2,24,25], and the basic estimates of related widths of finite dimensional spaces due to Kashin [12], Gluskin [8] and Edmunds and Triebel [4] with [5,6,21,32].