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On the L^1 extremal problem for entire functions

P. Yuditskii

Abteilung für Dynamische Systeme und Approximationstheorie, Johannes Kepler Universität Linz, A-4040 Linz, Austria

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Abstract

We generalize the Korkin–Zolotarev theorem to the case of entire functions having the smallest L^1 norm on a system of intervals E. If $\mathbb{C} \setminus E$ is a domain of Widom type with the Direct Cauchy Theorem, we give an explicit formula for the minimal deviation. Important relations between the problem and the theory of canonical systems with reflectionless resolvent functions are shown.

Keywords: Korkin–Zolotarev theorem; de Branges spaces; Martin function; Widom domains; Approximation by entire functions; Canonical systems; Spectral theory

1. Introduction

The classical L^2, L^1, and L^∞ extremal problems for polynomials and entire functions have numerous connections with diverse problems in analysis: the investigations of Abel on expressing elliptic and hyperelliptic integrals in terms of elementary functions; continued fractions and orthogonal polynomials; spectral properties of periodic (and almost periodic) Jacobi matrices and Schrödinger operators; factorization of functions on Riemann surfaces; subharmonic majorants; special conformal mappings onto "comb-like domains"; Pell's equations and so on.

Probably the earliest version of the results we are interested in is the following Korkin–Zolotarev theorem (although implicitly it was already in [36]).

E-mail address: Petro.Yudytskiy@jku.at.

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Theorem 1.1. *Let $\mathcal{P}_n$ be the set of polynomials of degree at most n. Then*

$$M := \inf_{P(x)=x^n+\cdots\in\mathcal{P}_n} \int_{-1}^{1} |P(x)|dx = \frac{1}{2^{n-1}} \tag{1.1}$$

and the extremal polynomial $U_n(x)$ is the Chebyshev polynomial of the second kind, i.e.,

$$U_n(x) = \frac{1/\zeta^{n+1} - \zeta^{n+1}}{2^n(1/\zeta - \zeta)}, \quad x = \frac{1}{2}(1/\zeta + \zeta).$$

Various generalizations were given by Stieltjes, Markov, Posse, Bernstein, Akhiezer and Krein. The following two directions are the most important ones for us: we would like to pass from the polynomial case to classes of entire functions and from a single interval of integration to a system of intervals, ideally to an arbitrary closed subset of $\mathbb{R}$.

A two interval version of the Korkin–Zolotarev problem was investigated by N.I. Akhiezer [2]. The several interval case was studied by Akhiezer and Krein in the following general setting [20, Chapter VIII, Sect. 9].

Problem 1.2. Let E be a system of intervals, $E = [b_0, a_0]\setminus\cup_{j=1}^{m}(a_j, b_j)$. For a given real vector $\{\lambda_k\}_{k=0}^{n}$, $\sum\lambda_k^2 > 0$, define the following functional on $\mathcal{P}_n$

$$\Lambda(P) = \sum \lambda_k c_k, \qquad P(x) = \sum c_k x^k.$$

Find

$$M = M(\Lambda) = \inf_{P\in\mathcal{P}_n, \Lambda(P)=1} \int_E |P(x)|dx.$$

Of course, very interesting partial cases of the problem correspond to the choices $\Lambda(P) = \Lambda_{x_0}(P) = P(x_0)$, $x_0 \in \mathbb{R}\setminus E$, and Λ_∞ when $\lambda_k = 0$ for every $0 \le k < n$ and $\lambda_n = 1$, see (1.1).

Due to the duality principle [5] this problem is equivalent to the so-called Markov's L moment problem, namely, $M(\Lambda)$ is equal to $1/L(\Lambda)$ in the following problem.

Problem 1.3. Find the smallest $L = L(\Lambda) > 0$ such that the moment problem

$$\int_E x^k f(x)dx = \lambda_k$$

is solvable with a real function $f(x)$ and $|f(x)| \le L$, $x \in E$.

Problem 1.3 was investigated in [3]. One has to consider 2^m sequences $\{s_k\}_{k=0}^{n+1} = \{s_k(\delta_1, \dots, \delta_m)\}_{k=0}^{n+1}$, $\delta_j = \pm 1$, given by the expansions

$$\frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_{n+1}}{z^{n+2}} + \cdots = \sqrt{\frac{z-a_0}{z-b_0}} \prod_{j\ge 1} \sqrt{\frac{z-a_j}{z-b_j}}^{\,\delta_j} e^{\frac{1}{2L}\left(\frac{\lambda_0}{z} + \frac{\lambda_1}{z^2} + \cdots + \frac{\lambda_n}{z^{n+1}} + \cdots\right)}. \tag{1.2}$$

Then L is the smallest value for which all $\{s_k\}_{k=0}^{n+1}$'s are moments of positive measures supported on $[b_0, a_0]$. There are well known algebraic conditions for a sequence of numbers to be moments of a positive measure given in terms of positivity of the associated Hankel matrices, so one gets a system of algebraic relations for L.