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On the generalized Askey–Wilson polynomials

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Abstract

In this paper, a generalization of Askey–Wilson polynomials is introduced. These polynomials are obtained from the Askey–Wilson polynomials via the addition of two mass points to the weight function of them at the points ± 1. Several properties of such new family are considered, in particular, the three-term recurrence relation and the representation as basic hypergeometric series.

Keywords: Krall-type polynomials; Askey–Wilson polynomials; Second order linear difference equation; q-polynomials; Basic hypergeometric series

1. Introduction

The Krall-type polynomials are orthogonal with respect to a linear functional $\widetilde{\mathbf{u}}$ obtained from a quasi-definite functional $\mathbf{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures. These polynomials appear as eigenfunctions of a fourth order linear differential operator with polynomial coefficients that do not depend on the degree of the polynomials. They were first considered by Krall in [21] (for a more recent reviews see [4] and [22, Chapter XV]). In fact, Krall discovered that there are only three extra families of orthogonal polynomials apart from the classical polynomials of Hermite, Laguerre and Jacobi that satisfy such a fourth order differential equation which are orthogonal

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with respect to measures that are not absolutely continuous with respect to the Lebesgue measure. This result motivated the study of the polynomials orthogonal with respect to the more general weight functions [18,20] that could contain more instances of orthogonal polynomials being eigenfunctions of higher-order differential equations [22, Chapters XVI, XVII].

In the last years the study of such polynomials have been considered by many authors (see e.g. [2,5,15,16,23,26] and the references therein) with a special emphasis on the case when the starting functional $\mathbf{u}$ is a classical continuous, discrete or q-linear functional with the linear type lattices (for more details see [3,5] and references therein). In fact, for the q-case some examples related with the q-Laguerre and the little q-Jacobi polynomials were constructed by Haine and Grünbaum in [15] using the Darboux transformation. Later on, in [26], Vinet and Zhedanov presented a more complete study for the little q-Jacobi polynomials. In these both cases, the q-Krall polynomials satisfy a higher order q-difference equations with polynomial coefficients independent of n. For the discrete case the problem was solved very recently by Durán using a new method (see [13,12], for details).

For the general q-quadratic lattice only few results were known. An important contribution to this case was done in [17] where the authors considered a generalized Askey–Wilson polynomials by adding mass points. They showed that the resulting orthogonal polynomials satisfy a higher order q-difference equation with polynomial coefficients independent of n, only if the masses are added at very specific points out of the interval of orthogonality $[-1, 1]$. Another contribution to this problem was done in [6], where a general theory of the Krall-type polynomials on non-uniform lattices was developed. In fact, in [6] the authors studied the polynomials $\widetilde{P}_n(s)_q$ which are orthogonal with respect to the linear functionals $\widetilde{\mathbf{u}} = \mathbf{u} + \sum_{k=1}^{N} A_k \delta_{x_k}$ defined on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ and considered, as a representative example, the Krall-type Racah polynomials (see also [7]). In fact, in [6, Section 5], we posed the problem of obtaining a generalization of the Askey–Wilson polynomials by adding two mass points at the end of the interval of orthogonality, motivated by the results in [17].

Thus our main aim here is to study the orthogonal polynomials obtained via the addition of two mass points at the end of the interval of orthogonality of the Askey–Wilson polynomials. The structure of the paper is as follows. In Section 2, some preliminary results on the Askey–Wilson polynomials are presented as well as the most general expression for the kernels on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. Our main results are in Section 3, where we introduce a detailed study of the generalized Askey–Wilson polynomials obtained from the classical Askey–Wilson polynomials by adding two mass points at ∓ 1.

2. Preliminary results

Here we include some results of the theory of orthogonal polynomials on the non-uniform lattice (for more details see e.g., [1,24])

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3. \tag{1}$$

The polynomials on non-uniform lattices $P_n(s)_q := P_n(x(s))$ are the polynomial solutions of the second order linear difference equation (SODE) of hypergeometric type

$$A_s y(s+1) + B_s y(s) + C_s y(s-1) + \lambda_n y(s) = 0,$$

$$A_s = \frac{\sigma(s) + \tau(s)\Delta x\left(s - \frac{1}{2}\right)}{\Delta x(s)\Delta x\left(s - \frac{1}{2}\right)}, \qquad C_s = \frac{\sigma(s)}{\nabla x(s)\Delta x\left(s - \frac{1}{2}\right)}, \qquad B_s = -A_s - C_s, \tag{2}$$