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Polynomial Schauder basis of optimal degree with Jacobi orthogonality

Jürgen Prestin, Jörn Schnieder*

Institut für Mathematik, Universität zu Lübeck, Ratzeburger Allee 160, 23562 Lübeck, Germany

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Abstract

In our paper we construct a polynomial Schauder basis $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$ of optimal degree with Jacobi orthogonality. A candidate for such a basis is given by the use of some wavelet theoretical methods, which were already successful in the case of Tchebysheff and Legendre orthogonality. To prove that this sequence is in fact a Schauder basis for $C[-1,1]$ and as the main difficulty of the whole proof we show the uniform boundedness of its Lebesgue constants

$$\sup_{x\in[-1,1],n\in\mathbb{N}_0}\left\|\sum_{j=0}^{n}p_{\alpha,\beta,j}(x)p_{\alpha,\beta,j}\right\|_{L^1_{\omega_{\alpha,\beta}}[-1,1]}<\infty.$$

Keywords: Schauder basis; Approximation by polynomials; Asymptotic approximation; Jacobi polynomials; Bessel functions; Hypergeometric functions

1. Introduction

In this paper our main goal is to construct a Schauder basis for the Banach space $(C[-1,1], \|\cdot\|_\infty)$ whose elements consist of algebraic polynomials, are of minimal degree and orthogonal

* Corresponding author.
E-mail addresses: prestin@math.uni-luebeck.de (J. Prestin), schniede@math.uni-luebeck.de (J. Schnieder).

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with respect to the weighted inner product $\langle f, g\rangle_{\omega_{\alpha,\beta}} := \langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\omega_{\alpha,\beta}(x)\mathrm{d}x$, where $\omega_{\alpha,\beta}$ with $\alpha, \beta \geq -\frac{1}{2}$ and $\max\{\alpha, \beta\} > -\frac{1}{2}$ is the usual Jacobi weight with

$$\omega_{\alpha,\beta}(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta} & \text{for all } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

As usual we define a Schauder basis to be a sequence of functions $(p_n)_{n\in\mathbb{N}_0} \subseteq C[-1,1]$ such that there exists for every $f \in C[-1,1]$ a uniquely determined sequence $(c_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$ with $f = \sum_{n=0}^{\infty} c_n p_n$ with respect to the $\|\cdot\|_\infty$-norm on $[-1,1]$. If, in addition, the functions p_n are polynomials, then we call the sequence $(p_n)_{n\in\mathbb{N}_0}$ a polynomial Schauder basis. A polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ is said to be optimal for a given $\varepsilon > 0$ if

$$\deg p_n \leq (1+\varepsilon)n \quad \text{for all } n \in \mathbb{N}_0,$$

where $\deg p_n$ denotes the degree of the polynomial p_n.

Optimal polynomial bases have been an object of intensive discussions in numerical and applied mathematics for over 100 years. For example one of the first famous but negative results was given in 1914 by Faber [4], who showed that there is no polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ with $\deg p_n = n$ for every $n \in \mathbb{N}_0$. In Ul'yanov's formulation the problem was posed again in [27], namely to find a minimal possible growth of degrees of polynomials forming a Schauder basis or an orthogonal Schauder basis for $C[a,b]$. The breakthrough in this area of research was achieved by Privalov in 1987 and 1990: in [17] Privalov showed that for any polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ (orthogonal or not) there exists $\varepsilon > 0$ such that $\max_{j\leq n} \operatorname{grad} p_j \geq (1+\varepsilon)n$ for all sufficiently large n. In [18] Privalov succeeded in showing that the aforementioned result is optimal in the sense that for any $\varepsilon > 0$ there exists a (not necessarily orthogonal) optimal polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ for $C[-1,1]$, i.e. a polynomial basis with $\deg p_n \leq (1+\varepsilon)n$ for all sufficiently large $n \in \mathbb{N}$. Since that time the research in this area increased rapidly (compare for example Offin and Oskolkov in [12], Privalov in [19] and Ul'yanov in [28]). With these results in mind there was also progress in a more general problem, namely to construct an optimal Schauder basis with a given orthogonality relation for the basis functions: for the trigonometric case $C_{2\pi}$, Lorentz and Sahakian solved the problem in 1994 (see [9,16]), for the algebraic case $C[-1,1]$ with the four Tchebysheff weights (i.e. $|\alpha| = |\beta| = \frac{1}{2}$ in (1)), that means in a situation rather similar to the trigonometric case, the problem was solved by us in 1996 and 2000 in [8] and [6] respectively. In 2003 Khabiboulline generalized these results in [7] for the Tchebysheff weights multiplied with a Szegö weight. Finally, in 1999 Skopina in [23–25] and a little bit later and independently Woźniakowski succeeded in [29] in solving the problem for the Legendre weight (i.e. $\alpha = \beta = 0$ in (1)) (and thus solving Ul'yanov's problem at all), which was in some sense the simplest non-trigonometric case for the class of Jacobi weights. Now, the main goal of this paper is to generalize the aforementioned results to the case of Jacobi orthogonality.

The general idea is to use wavelet-like constructions which were already successful in cases of the Tchebysheff and Legendre weights. Therefore, we look for polynomials which could be interpreted as wavelet-like or scaling-like functions, of the form

$$\psi_k^{(\alpha,\beta)} := M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} g\left(\frac{s}{M}\right) \cos((3M+s)\theta_k)\, p_{N-M+s}^{(\alpha,\beta)} \quad \text{for } k = 0, \ldots, 2M-1,$$