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Generating functions of orthogonal polynomials in higher dimensions

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Abstract

In this paper two important classes of orthogonal polynomials in higher dimensions using the framework of Clifford analysis are considered, namely the Clifford–Hermite and the Clifford–Gegenbauer polynomials. For both classes an explicit generating function is obtained.

Keywords: Orthogonal polynomials; Cauchy–Kowalevski extension theorem; Fueter's theorem; Clifford–Hermite polynomials; Clifford–Gegenbauer polynomials

1. Introduction

Clifford analysis is a refinement of harmonic analysis: it is concerned among others, with the study of functions in the kernel of the Dirac operator, a first order operator that squares to the Laplace operator. In its study, special sets of orthogonal polynomials play an important role, see e.g. [7] for applications to generalized Fourier transforms and [3] for wavelet transforms. In particular the so-called Clifford–Hermite and Clifford–Gegenbauer polynomials [22] have received considerable attention. Our main aim in this paper is to find explicit generating functions for both sets of polynomials.

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Let us start by introducing the necessary notations and definitions. We denote by $\mathbb{R}_{0,m}$ the real Clifford algebra generated by the standard basis $\{e_1, \ldots, e_m\}$ of the Euclidean space $\mathbb{R}^m$ (see [4]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m$$

and a general element $a \in \mathbb{R}_{0,m}$ may be written as

$$a = \sum_{A \subset M} a_A e_A, \quad a_A \in \mathbb{R},$$

where $e_A = e_{j_1} \ldots e_{j_k}$ for $A = \{j_1, \ldots, j_k\} \subset M = \{1, \ldots, m\}$, with $j_1 < \cdots < j_k$. For the empty set one puts $e_\emptyset = 1$, which is the identity element.

An important part of Clifford analysis is the study of so-called monogenic functions (see e.g. [2,9,11]). They are defined as follows. A function $f : \Omega \to \mathbb{R}_{0,m}$ defined and continuously differentiable in an open set Ω in $\mathbb{R}^{m+1}$ (resp. $\mathbb{R}^m$), is said to be monogenic if

$$(\partial_{x_0} + \partial_{\underline{x}}) f = 0 \quad (\text{resp. } \partial_{\underline{x}} f = 0) \text{ in } \Omega,$$

where $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$ is the Dirac operator in $\mathbb{R}^m$. Note that the differential operator $\partial_{x_0} + \partial_{\underline{x}}$, called generalized Cauchy–Riemann operator, provides a factorization of the Laplacian, i.e.

$$\Delta = \sum_{j=0}^{m} \partial_{x_j}^2 = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}).$$

Thus monogenic functions, like their lower dimensional counterpart (i.e. holomorphic functions), are harmonic.

In this paper we shall deal with two fundamental techniques to generate special monogenic functions: the Cauchy–Kowalevski extension theorem (see [2,9,21]) and Fueter's theorem (see [10,18,20,23]).

The CK-extension theorem states that: *Every $\mathbb{R}_{0,m}$-valued function $g(\underline{x})$ analytic in the open set $\underline{\Omega} \subset \mathbb{R}^m$ has a unique monogenic extension given by*

$$\mathsf{CK}[g(\underline{x})](x_0, \underline{x}) = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!} \partial_{\underline{x}}^n g(\underline{x}), \tag{1}$$

and defined in a normal open neighborhood $\Omega \subset \mathbb{R}^{m+1}$ of $\underline{\Omega}$.

Fueter's theorem discloses a remarkable connection existing between holomorphic functions and monogenic functions. It was first discovered by R. Fueter in the setting of quaternionic analysis (see [10]) and later generalized to higher dimensions in [18,20,23]. For further works on this topic we refer the reader to e.g. [5,6,12,14–17,19].

Throughout the paper we assume $P_k(\underline{x})$ to be a given arbitrary homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$. In this paper we are concerned with the following generalization of Fueter's theorem obtained in [23].

Let $h(z) = u(x, y) + i v(x, y)$ be a holomorphic function in some open subset Ξ of the upper half of the complex plane $\mathbb{C}$. Put $\underline{\omega} = \underline{x}/r$, with $r = |\underline{x}|$, $\underline{x} \in \mathbb{R}^m$. If m is odd, then the function

$$\mathsf{Ft}\left[h(z), P_k(\underline{x})\right](x_0, \underline{x}) = \Delta^{k+\frac{m-1}{2}} \left[\left(u(x_0, r) + \underline{\omega}\, v(x_0, r)\right) P_k(\underline{x})\right]$$

is monogenic in $\Omega = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1} : (x_0, r) \in \Xi\}$.