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Infinite products of arbitrary operators and intersections of subspaces in Hilbert space

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Abstract

Let m closed linear subspaces $S_1, \ldots, S_m$ of a given Hilbert space H and m sequences of arbitrary operators $\left(A_n^{(k)}\right)_{n=1}^{\infty}$, $k = 1, \ldots, m$, be connected by the relations $\lim_{n\to\infty} \|A_n^{(k)} x - P_{S_k} x\| = 0$ for each $x \in H$, $k = 1, \ldots, m$, where P denotes the orthogonal projection onto the corresponding subspace. We derive sufficient conditions on the operators $A_n^{(k)}$, which yield strong convergence of the infinite products $\prod_{n=1}^{\infty}\left(A_n^{(m)} \cdots A_n^{(1)}\right) x$ for any $x \in H$, with the limits belonging to the intersection of all the subspaces $S_1, \ldots, S_m$. Several counterexamples show the optimality of our conditions.

Keywords: Fixed point; Halperin's theorem; Hilbert space; Infinite product; Orthogonal projection

1. Introduction

Let $S_1, S_2, \ldots, S_m$ be m closed linear subspaces of a given Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with induced norm $\|\cdot\|$ and let S be their intersection. A celebrated theorem, established by I. Halperin [4] in 1962, declares that, *for each* $x \in H$, *one has*

$$\lim_{n\to\infty} \|(P_{S_m} P_{S_{m-1}} \cdots P_{S_1})^n x - P_S x\| = 0, \tag{1.1}$$

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where the letter P stands for the orthogonal projection onto the corresponding subspace (the case $m = 2$ was proved much earlier by J. von Neumann [9]).

Note that the convergence in (1.1) need not be uniform (on bounded sets) and may even be arbitrarily slow. This prevents Halperin's result from being generalized to other kinds of operator products without additional conditions on the m-tuple $(S_1, \ldots, S_m)$, conditions which lead to appropriate norm estimates for the product $P_{S_m} P_{S_{m-1}} \cdots P_{S_1}$. For example, in the papers [5,6] we required positivity of the inclination $\ell(S_1, S_2, \ldots, S_m)$. Only in this way we could prove convergence of some non-periodic infinite products of projection operators and even insert into these products additional nonexpansive, possibly nonlinear operators. However, the additional operators did not provide any input into this convergence, which was only implied by special estimates of the inclination and was always uniform for any bounded set of the initial points of the iterations.

In what follows we try to completely *exclude* the projection operators, *replacing* them with arbitrary, possibly nonlinear operators, but still obtaining the same conclusion as in Halperin's theorem, that is, convergence to the intersection S without obligatory uniformity. We also give up any common properties of the subspaces $S_1, \ldots, S_m$ such as positivity of the inclination or some special angles between these subspaces. Of course, in order to achieve this goal, we have to postulate some connection between the operators and the subspaces. For instance, replacing for $m = 2$ the projections P_{S_1} and P_{S_2} by arbitrary operators A and B, we may require that, for any $x \in H$, the sequence $(A^n x)_{n=1}^{\infty}$ strongly converges to $P_{S_1} x$ and the sequence $(B^n x)_{n=1}^{\infty}$ strongly converges to $P_{S_2} x$. The desired result should be that the sequence $((AB)^n x)_{n=1}^{\infty}$ strongly converges to some point $\bar{x} \in S_1 \cap S_2$ (which may, however, be different from $P_{S_1 \cap S_2} x$).

Unfortunately, such a result is not true in general. A counterexample may be constructed in a way similar to the one we used in [7]. Namely, let $H = \ell_2$ and let S_1, S_2 be the one-dimensional subspaces spanned by the vectors $\mathbf{e}_1 = (1, 0, 0, \ldots)$ and $\mathbf{e}_2 = (0, 1, 0, \ldots)$, respectively. Now, for any $\mathbf{x} = (x_1, x_2, \ldots) \in \ell_2$, we define the operators A and B by

$$A\mathbf{x} := (x_1, 0, 0, \alpha_1 x_3, \beta_1 x_4, \ldots, \alpha_i x_{2i+1}, \beta_i x_{2i+2}, \ldots)$$

and

$$B\mathbf{x} := (0, x_2, 0, \beta_1 x_3, \alpha_1 x_4, \ldots, \beta_i x_{2i+1}, \alpha_i x_{2i+2}, \ldots),$$

where $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_i\}_{i=1}^{\infty}$ are two sequences of positive real numbers such that all $\alpha_i, \beta_i < 1$, and

$$\prod_{i=1}^{\infty} \alpha_i = a > 0, \qquad \prod_{i=1}^{\infty} \beta_i = 0.$$

These operators are obviously nonexpansive and

$$\lim_{n\to\infty} \|A^n \mathbf{x} - P_{S_1} \mathbf{x}\| = \lim_{n\to\infty} \|B^n \mathbf{x} - P_{S_2} \mathbf{x}\| = 0$$

for any $\mathbf{x} \in \ell_2$. Indeed, given $\mathbf{x} \in \ell_2$, we get $\|A^{2n}\mathbf{x} - P_{S_1}\mathbf{x}\|^2 = s_n^{(1)} + s_n^{(2)}$, where

$$s_n^{(1)} = \sum_{i=1}^{\infty} \left(x_{2i+1} \prod_{j=i}^{i+n-1} \alpha_j \beta_j \right)^2, \qquad s_n^{(2)} = \sum_{i=1}^{\infty} \left(x_{2i+2} \prod_{l=i+1}^{i+n} \alpha_l \prod_{j=i}^{i+n-1} \beta_j \right)^2.$$