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Journal of Approximation Theory 177 (2014) 1–16

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Best m-term trigonometric approximation of periodic functions of several variables from Nikol'skii–Besov classes for small smoothness

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Received 8 November 2011; accepted 18 September 2013
Available online 7 October 2013

Communicated by Paul Nevai

Abstract

We find the asymptotic orders of best m-term trigonometric approximation for Nikol'skii–Besov classes $\mathbf{B}^r_{p,\theta}$ of periodic functions of several variables in the space L_q, $1 \le p \le 2 < q < \infty$, for the cases of "small" smoothness $\frac{d}{p} - \frac{d}{q} < r < \frac{d}{p}$ and for the "critical" value of the smoothness $r = \frac{d}{p}$. These results are complementary to the estimates of the best m-term trigonometric approximation for classes $\mathbf{B}^r_{p,\theta}$ obtained by R.A. DeVore and V.N. Temlyakov.

Keywords: Best m-term trigonometric approximation; Nikol'skii–Besov class; Small smoothness; Greedy approximation

1. Introduction and main results

Let $\mathbb{R}^d$, $d \ge 1$, be the d-dimensional Euclidean space of points $\mathbf{x} = (x_1, \dots, x_d)$, and $L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, $\mathbb{T}^d = \prod_{j=1}^d [-\pi; \pi)$, be the space of functions $f(\mathbf{x}) = f(x_1, \dots, x_d)$ that are 2π-periodic in each variable and their norm

$$\|f\|_p = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

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0021-9045/$ - see front matter
http://dx.doi.org/10.1016/j.jat.2013.09.006

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{T}^d} |f(\mathbf{x})|,$$

is finite.

Furthermore, let $k \in \mathbb{N}$ and $\mathbf{h} \in \mathbb{R}^d$. For $f \in L_p(\mathbb{T}^d)$, we denote

$$\triangle_{\mathbf{h}} f(\mathbf{x}) = f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}),$$

and we define the difference of order k with a step $\mathbf{h}$ by the following formula:

$$\Delta^k_{\mathbf{h}} f(\mathbf{x}) = \Delta_{\mathbf{h}}\Delta^{k-1}_{\mathbf{h}} f(\mathbf{x}), \qquad \Delta^0_{\mathbf{h}} f(\mathbf{x}) = f(\mathbf{x}).$$

The modulus of smoothness of order k for a function $f \in L_p(\mathbb{T}^d)$ is defined by the formula

$$\omega_k(f,t)_p = \sup_{|\mathbf{h}|\le t} \|\Delta^k_{\mathbf{h}} f\|_p,$$

where $|\mathbf{h}| = \sqrt{h_1^2 + \cdots + h_d^2}$.

We say that a function $f \in L_p(\mathbb{T}^d)$ belongs to the space $B^r_{p,\theta}$, $1 \le p, \theta \le \infty, r > 0$, if

$$\left(\int_0^\infty (t^{-r}\omega_k(f,t)_p)^\theta \,\frac{dt}{t}\right)^{1/\theta} < \infty, \quad 1 \le \theta < \infty,$$

and

$$\sup_{t>0} t^{-r}\omega_k(f,t)_p < \infty, \quad \theta = \infty.$$

The norm of the space $B^r_{p,\theta}$ is defined by formulas

$$\|f\|_{B^r_{p,\theta}} = \|f\|_p + \left(\int_0^\infty (t^{-r}\omega_k(f,t)_p)^\theta \,\frac{dt}{t}\right)^{1/\theta}, \quad 1 \le \theta < \infty, \tag{1.1}$$

and

$$\|f\|_{B^r_{p,\infty}} = \|f\|_p + \sup_{t>0} t^{-r}\omega_k(f,t)_p, \quad \theta = \infty, \tag{1.2}$$

for some $k > r$. It is known that the above definition is independent (up to norm equivalence) of the choice of the integer $k > r$ (see, e.g., [6]).

The spaces $H^r_p \equiv B^r_{p,\infty}$ and $B^r_{p,\theta}$, $1 \le \theta < \infty$, were introduced by S.M. Nikol'skii [26] and O.V. Besov [3] respectively.

The Nikol'skii–Besov class

$$\mathbf{B}^r_{p,\theta} := \{f \in B^r_{p,\theta} : \|f\|_{B^r_{p,\theta}} \le 1\}$$

is defined as the unit ball of the space $B^r_{p,\theta}$.

Note that the classes $\mathbf{B}^r_{p,\theta}$ and $\mathbf{H}^r_p$ were studied from the approximation viewpoint in [4,5,7,12,8,36,15,29,30,32,11], where additional relevant references can be found.

Now we define approximation characteristics that will be investigated in the present paper.

Let Θ_m be a set of m arbitrary d-dimensional vectors with integer coordinates and $P(\Theta_m,\mathbf{x}) = \sum_{\mathbf{k}\in\Theta_m} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$, $(\mathbf{k},\mathbf{x}) = k_1x_1 + \cdots + k_dx_d$, be trigonometric polynomials with