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Closed form representations and properties of the generalised Wendland functions

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Abstract

In this paper we investigate the generalisation of Wendland's compactly supported radial basis functions to the case where the smoothness parameter is not assumed to be a positive integer or half-integer and the parameter ℓ, which is chosen to ensure positive definiteness, need not take on the minimal value. We derive sufficient and necessary conditions for the generalised Wendland functions to be positive definite and deduce the native spaces that they generate. We also provide closed form representations for the generalised Wendland functions in the case when the smoothness parameter is an integer and where the parameter ℓ is any suitable value that ensures positive definiteness, as well as closed form representations for the Fourier transform when the smoothness parameter is a positive integer or half-integer.

1. Generalised Wendland functions

Positive definite functions are frequently found at the heart of scattered data fitting algorithms both in Euclidean space and on spheres: see [17]. The aim of this paper is to investigate a large class of such functions. The following definition fixes the notation for what follows.

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Definition 1.1. A function $\phi : [0, \infty) \to \mathbb{R}$ is said to generate a positive definite radial function on $\mathbb{R}^d$, if, for any $n \geq 2$ distinct locations $x_1, \ldots, x_n \in \mathbb{R}^d$, the following $n \times n$ distance matrix

$$\Big(\phi(\|x_j - x_k\|)\Big)_{j,k=1}^{n}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm, is positive definite.

For such functions we have the following characterisation theorem (see [5] p. 34).

Theorem 1.2. *A continuous function $\phi : [0, \infty) \to \mathbb{R}$ such that $r \mapsto r^{d-1}\phi(r) \in L_1[0, \infty)$ generates a positive definite radial function on $\mathbb{R}^d$ if and only if the d-dimensional Fourier transform*

$$\mathcal{F}_d\phi(z) = z^{1-\frac{d}{2}} \int_0^\infty \phi(y)\, y^{\frac{d}{2}}\, J_{\frac{d}{2}-1}(yz)\, dy, \tag{1.2}$$

(where $J_\nu(\cdot)$ denotes the Bessel function of the first kind with order ν) is non-negative and not identically equal to zero.

In this paper we will investigate a family of parameterised basis functions that is generated by a truncated power function. Specifically, we set

$$\phi_{\mu,0}(r) := (1-r)_+^\mu = \begin{cases} (1-r)^\mu & \text{for } 0 \leq r \leq 1; \\ 0 & \text{for } r \geq 0, \end{cases}$$

and consider

$$\phi_{\mu,\alpha}(r) := \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_r^1 \phi_{\mu,0}(t)\, t \left(t^2 - r^2\right)^{\alpha-1} \mathrm{d}t \quad \text{for } r \in [0, 1], \tag{1.3}$$

where $\mu > -1$, $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function.

Before we embark on our investigation we briefly review what is already known of this family. Firstly, it is well known (see [7]) that if $\alpha = k \in \{0, 1, 2, \ldots\}$ then the function $\phi_{\mu,k}$ generates a positive definite function on $\mathbb{R}^d$ if and only if

$$\mu \geq \frac{d+1}{2} + k. \tag{1.4}$$

In particular, quoting [7], the function $\phi_{\mu,k}$ is $2k$ times differentiable at zero, positive, strictly decreasing on its support and has the form

$$\phi_{\mu,k}(r) = p_k(r)(1-r)_+^{\mu+k}, \tag{1.5}$$

where p_k is a polynomial of degree k with coefficients in μ and $(x)_+ := \max(x, 0)$. In [16] Wendland considers the case where

$$\mu = \ell := \left\lfloor \frac{d}{2} \right\rfloor + k + 1 \tag{1.6}$$

i.e., the smallest allowable integer that still allows positive definiteness. In this setting we can deduce from (1.5) that $\phi_{\ell,k}$ is a polynomial of degree $2k + \ell$ on the unit interval. Furthermore, Wendland has shown (see Chapter 10 of [17]) that, when d is odd, the function

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi_{\frac{d+1}{2}+k,k}(\|\mathbf{x} - \mathbf{y}\|), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{1.7}$$