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Multivariate polynomial interpolation on lower sets

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Abstract

In this paper we study multivariate polynomial interpolation on lower sets of points. A lower set can be expressed as the union of blocks of points. We show that a natural interpolant on a lower set can be expressed as a linear combination of tensor-product interpolants over various intersections of the blocks that define it.

Keywords: Multivariate polynomial interpolation; Lower sets; Tensor-product interpolants

1. Introduction

Lower sets are subsets of Cartesian grids of points in $\mathbb{R}^d$ that admit unique polynomial interpolation from a natural, associated space of polynomials, and include rectangular and triangular grids as special cases. Such interpolation has been studied, for example, in [1,3–6, 8–13]. This natural polynomial space is the span of a collection of monomials with powers taken from a similar lower set of indices. The resulting interpolant is the "least interpolant", introduced in [3], and studied further in [2].

A lower set of points can be expressed as the union of blocks (rectangular grids) of points. The purpose of this paper is (1) to make the observation that the interpolant can be expressed as a linear combination of tensor-product interpolants, each corresponding to an intersection of some

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of these blocks, and (2) to give an explicit formula for the linear combination, and to show how the formula simplifies in various important cases, including the 2D case, and the total degree case. It was brought to our attention, after submitting the paper, that similar relations were obtained for piecewise polynomial interpolants on sparse grids, for the numerical solution of PDE's. In particular, our formula for the total degree polynomial interpolant is similar to the 'combination technique' for sparse grids (see e.g. [7]).

Here is an outline of the paper. In Section 2 we introduce some notation, and discuss tensor-product polynomial interpolation. In Section 3 we present the interpolation problem on lower sets, and give an explicit expression for the interpolants in terms of a Newton-type formula. An incremental double sum formula for these interpolants, based on blocks, is derived in Section 4, while its simplifications are obtained in Section 5 for the 2D case. For the arbitrary dimension case, a general formula in terms of tensor-product interpolants is derived in Section 6, and again simplified. The simplified formula is applied in Section 7 to interpolation by polynomials of a fixed total degree.

2. Tensor-product interpolation

We consider a Cartesian grid of points $\mathcal{X} \subset \mathbb{R}^d$. For each $j \in \{1, \ldots, d\}$, let $x_{j,k}$, $k \in \mathbb{N}_0$, be a sequence of distinct real points. Then the points of $\mathcal{X}$ are

$$x_\alpha = (x_{1,\alpha_1}, x_{2,\alpha_2}, \ldots, x_{d,\alpha_d}), \quad \alpha \in \mathbb{N}_0^d,$$

where we use the multi-index notation

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d,$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_d$, and the convention that $\alpha \le \beta$ means that $\alpha_j \le \beta_j$ for $j = 1, \ldots, d$. Given any $\beta \in \mathbb{N}_0^d$, we call the set of indices

$$B_\beta = \{\alpha \in \mathbb{N}_0^d : 0 \le \alpha \le \beta\},$$

a *block*, and with it we can associate the set of grid points

$$X_\beta = \{x_\alpha : \alpha \in B_\beta\}.$$

Note that unlike B_β, the points of X_β are not necessarily consecutive in the grid $\mathcal{X}$, as illustrated in Fig. 1, showing an example of the set $X_{(1,1)}$ corresponding to the block $B_{(1,1)}$.

If we denote a monomial in $\mathbb{R}^d$ by

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d},$$

for $\alpha \in \mathbb{N}_0^d$, then

$$P_\beta = \operatorname{span}\{x^\alpha : \alpha \in B_\beta\}$$

is an interpolation space for X_β. In other words, for every function $f : \mathbb{R}^d \to \mathbb{R}$ there is a unique tensor-product polynomial $p \in P_\beta$ such that $p(x_\alpha) = f(x_\alpha)$ for all $\alpha \in B_\beta$.

One way of expressing p is the Newton form; see [8], Chap. 5. We define, for each $j = 1, \ldots, d$, the univariate polynomials $\omega_{j,0}(y) = 1$ and

$$\omega_{j,k}(y) = \prod_{i=0}^{k-1} (y - x_{j,i}), \quad k \ge 1, \; y \in \mathbb{R},$$