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# How poles of orthogonal rational functions affect their Christoffel functions

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**Abstract**

We show that even a relatively small number of poles of a sequence of orthogonal rational functions approaching the interval of orthogonality, can prevent their Christoffel functions from having the expected asymptotics. We also establish a sufficient condition on the rate for such asymptotics, provided the rate of approach of the poles is sufficiently slow. This provides a supplement to recent results of the authors where poles were assumed to stay away from the interval of orthogonality.


*Keywords:* Orthogonal rational functions; Christoffel functions

## 1. Introduction

Let $\mu$ be a finite positive Borel measure on $[-1, 1]$, with infinitely many points in its support. Then we can define orthonormal polynomials $p_n(x) = p_n(d\mu, x) = \gamma_n x^n + \cdots, n \geq 0$, satisfying

$$\int_{-1}^{1} p_n p_m d\mu = \delta_{mn}.$$

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We say the measure $\mu$ is *regular on* $[-1, 1]$ *in the sense of Stahl, Totik, and Ullmann*, or just *regular* [5], if

$$\lim_{n\to\infty} \gamma_n^{1/n} = 2.$$

An equivalent definition involves norms of polynomials of degree $\le n$:

$$\lim_{n\to\infty} \left[ \sup_{\deg(P)\le n} \frac{\|P\|^2_{L_\infty[-1,1]}}{\int_{-1}^{1} |P|^2 d\mu} \right]^{1/n} = 1.$$

Regularity of a measure is useful in studying asymptotics of orthogonal polynomials. One simple criterion for regularity is that $\mu' > 0$ a.e. on $[-1, 1]$, the so-called Erdős–Turán condition. However, there are pure jump measures, and pure singularly continuous measures that are regular.

We define the $n$th *Christoffel function* for $\mu$

$$\lambda_n(d\mu, x) = 1 \Big/ \sum_{j=0}^{n-1} p_j^2(d\mu, x),$$

which satisfies the extremal property

$$\lambda_n(d\mu, x) = \inf_{\deg(P)\le n-1} \frac{\int |P|^2 d\mu}{|P(x)|^2}.$$

A classical result of Maté et al. [4] (see also [6]) asserts that if $\mu$ is regular on $[-1, 1]$, and in some subinterval $[a, b]$

$$\int_a^b \log \mu' > -\infty,$$

then for a.e. $x \in [a, b]$,

$$\lim_{n\to\infty} n\lambda_n(d\mu, x) = \pi \mu'(x)\sqrt{1 - x^2}.$$

If instead we assume that $\mu$ is regular in $[-1, 1]$, while $\mu$ is absolutely continuous in a neighborhood of some $x \in (-1, 1)$, and $\mu'$ is continuous at $x$, then this last limit holds at $x$.

The aim of this paper is to further investigate asymptotic behavior of Christoffel functions, associated with orthogonal rational functions. The monograph [2] provides a comprehensive study of the theory of orthogonal rational functions.

We shall assume that we are given a sequence of extended complex numbers that will serve as our poles

$$A = \{\alpha_1, \alpha_2, \alpha_3, \ldots.\} \subset \bar{\mathbb{C}} \setminus [-1, 1].$$

We let $\pi_0(x) = 1$, and for $k \ge 1$,

$$\pi_k(x) = \prod_{j=1}^{k} (1 - x/\alpha_j).$$