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An algorithm to find a maximum of a multilinear map over a product of spheres

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Abstract

We provide an algorithm to compute the 2-norm maximum of a multilinear map over a product of spheres. As a corollary we give a method to compute the first singular value of a linear map and an application to the theory of entangled states in quantum physics. Also, we give an application to find a closest rank-one tensor of a given one.

Keywords: Maximum; Product of spheres; Algorithm; Multilinear map; First singular value

0. Introduction

Many problems in mathematics need to maximize a bilinear form over a product of spheres. For example, the 2-norm of a matrix is given by the maximum of the bilinear form $(x, y) \to x^t Ay$, where $\|x\| = \|y\| = 1$. Another interesting problem is to find a closest rank-one tensor of a given tensor, $\sum a_{ijk} x_i \otimes y_j \otimes z_k$. To answer this problem one has to find the maximum of a trilinear form over a product of three spheres (see the examples).

This article provides an algorithm to find the maximum of a multilinear map over a product of spheres,

$$\ell : \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_r+1} \to \mathbb{R}^{n_{r+1}+1}, \qquad \max_{\|x_1\|=\cdots=\|x_r\|=1} \|\ell(x_1, \ldots, x_r)\|.$$

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We have reduced the problem of finding the maximum of ℓ to a problem of finding fixed points of a map $\nabla\ell : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{r+1}} \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{r+1}}$. The advantage of this reduction is the possibility to count the number of extreme points of ℓ, and also, to find the fixed points of $\nabla\ell$ solving a system of polynomial equations. There are standard algebro-geometric tools to solve systems of polynomial equations.

In Section 1 we review some concepts and definitions in algebraic geometry such as projective spaces, maps, products of projective spaces and maps between them.

In Section 2, using Lagrange's method of multipliers, see [1, Section 13.7], we reduce the problem of finding the maximum of a multilinear map ℓ to the problem of finding fixed points of a map $\nabla\ell$. We compare our approach with the ones in the literature.

In Section 3 we make a digression to discuss the number of extreme points of a multilinear map over a product of spheres. We use intersection theory to count the number of fixed points of the map $\nabla\ell : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{r+1}} \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{r+1}}$. Recall that the number of fixed points of a generic map $F : \mathbb{P}^N \to \mathbb{P}^N$, of degree d, is $1 + d + \cdots + d^N$. In this section we give a formula to compute the number of extreme points of a multilinear map over a product of spheres. If the map is generic, this number is achieved over $\mathbb{C}$, and if it is not generic, this number is a bound when the extreme points are finite. In the literature, the extreme points of ℓ are called singular vectors (see [17]) and in this section we count them.

In Section 4 we use our approach to find the maximum of a bilinear form over a product of spheres. In the bilinear case, the map $\nabla\ell$, induces a linear map $L : \mathbb{P}^N \to \mathbb{P}^N$, where N is a natural number, and we prove that for a generic $q \in \mathbb{P}^N$, the sequence $\{q, L(q), L^2(q), \ldots\}$ converges to the absolute maximum. In other words, the absolute maximum is an attractive fixed point of L. Also, with the same tools, we give an algorithm to find the spectral radius of a square matrix.

In Section 5 we use the theory developed to present the algorithm. We take advantage of a result in Section 2; the classes of extreme points of a multilinear form ℓ, are in bijection with the fixed points of $\nabla\ell$. We reduce the problem of finding fixed points of $\nabla\ell$ to solve a system of polynomial equations with finitely many solutions. In the literature about computational aspects of algebraic geometry, there exists a lot of algorithms to solve a system of polynomial equations with finitely many solutions; see [10]. This gives us the ability to find the absolute maximum of ℓ. It is important to mention that the system of polynomial equations obtained with our approach is slightly different from the system of polynomial equations obtained naively from the method of Lagrange's multipliers. Our approach in projective geometry allows us to find the correct solution removing some constraints. In the first part of the section, we present a direct method to find the maximum value of a generic multilinear form over a product of spheres. Basically, it reduces to finding the spectral radius of a matrix. In the second part of the section, we give an algorithm to find the point $(x_1, \ldots, x_r) \in \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_r+1}$, where $\|x_i\| = 1, 1 \le i \le r$, such that $\|\ell(x_1, \ldots, x_r)\|$ is maximum.

In Section 6 we use the theory developed to compute a lot of examples and applications. One of them is the ability to find a closest rank-one tensor of a given tensor. We prove that this problem is well posed and we apply our algorithm to solve it. Another application is related to quantum physics. It is a criterion of separability. Given a quantum state, we can say if it is separable (see Remark 22 for definitions and related concepts).

1. Review on projective geometry

In this section we give some definitions that we are going to use such as projective spaces, maps, projective tangent spaces, product of projective spaces and maps between them. We are