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Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball

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Abstract

This work is a continuation of the recent study by the authors on approximation theory over the sphere and the ball. The main results define new Sobolev spaces on these domains and study polynomial approximations for functions in these spaces, including simultaneous approximation by polynomials and the relation between the best approximation of a function and its derivatives.

Keywords: Approximation by polynomials; Sobolev space; Modulus of smoothness; K-functional; Sphere; Ball

1. Introduction

In a recent work [3], the authors defined new moduli of smoothness and K-functionals on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ and the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \le 1\}$ of $\mathbb{R}^d$, where $\|x\|$ denotes the usual Euclidean norm, and used them to characterize the best approximation by polynomials. This work is a continuation of [3] and studies polynomial approximation in Sobolev spaces.

The new modulus of smoothness on the sphere is defined in terms of forward differences in the angle $\theta_{i,j}$ of the polar coordinates on the (x_i, x_j) planes, and it is essentially the maximum

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over all possible angles of the moduli of smoothness of one variable in $\theta_{i,j}$. There are $\binom{d}{2}$ such angles, which are clearly redundant as a coordinate system. Nevertheless, our new definition effectively reduces a large part of problems in approximation theory on $\mathbb{S}^d$ to problems on $\mathbb{S}^1$, which allows us to tap into the rich resources of trigonometric approximation theory for ideas and tools and adopt them for problems on the sphere. These same angles also become indispensable for our new definition of moduli of smoothness on the unit ball $\mathbb{B}^d$. In fact, our moduli on $\mathbb{B}^d$ are defined as the maximum of moduli of smoothness of one variable in these angles and of one additional term that takes care of the boundary behavior. We had two ways to define the additional term, the first one is deduced from the results on the sphere and the second one is the direct extension of the Ditzian–Totik modulus of smoothness on $[-1, 1]$, both of which capture the boundary behavior of the unit ball and permit both direct and inverse theorem for the best approximation. For $d = 1$, approximation by polynomials on $\mathbb{B}^1 = [-1, 1]$ is often deduced from approximation by trigonometric polynomials on the circle $\mathbb{S}^1$ by projecting even functions and their approximations on $\mathbb{S}^1$ to $[-1, 1]$. This procedure can be adopted to higher dimension by projection functions on $\mathbb{S}^d$ onto $\mathbb{B}^d$, and this is how our first modulus of smoothness on $\mathbb{B}^d$ was defined. It should be mentioned that our new moduli of smoothness on the sphere and on the ball are computable; in fact, the computation is not much harder than what is needed for computing classical modulus of smoothness of one variable. A number of examples were given in [3].

In the present paper we continue the work in this direction, study best approximation of functions and their derivatives. On the sphere, our result will be given in terms of differential operators $D_{i,j} = x_i \partial_j - x_j \partial_i$, which can be identified as partial derivatives with respect to $\theta_{i,j}$. We shall define Sobolev spaces and Lipschitz spaces in terms of $D_{i,j}$ on these two domains and study approximation by polynomials in these spaces, including simultaneous approximation of functions and their derivatives. The study is motivated by a question from Kendall Atkinson (cf. [1]) about the numerical solution of a Poisson equation.

The paper is organized as follows. The main results in [3] on the unit sphere will be recalled in Section 2 and the new results on the sphere will be developed in Section 3. The results in [3] on the unit ball will be recollected and further clarified in Section 4. Finally, the new results on the ball are developed in Section 5.

Throughout this paper we denote by $c, c_1, c_2, \ldots$ generic constants that may depend on fixed parameters and their values may vary from line to line. We write $A \lesssim B$ if $A \le cB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Polynomial approximation on the sphere: Recent progress

In this section we recall recent progress on polynomial approximation on the sphere as developed in [3].

Let $L^p(\mathbb{S}^{d-1})$ be the L^p-space with respect to the usual Lebesgue measure $d\sigma$ on $\mathbb{S}^{d-1}$ with norm denoted by $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{S}^{d-1})}$ for $1 \le p \le \infty$, where for $p = \infty$, we replace L^∞ by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the uniform norm.

2.1. Polynomial spaces and spherical harmonics

We denote by Π_n^d the space of polynomials of total degree n in d variables, and by $\Pi_n(\mathbb{S}^{d-1}) := \Pi_n^d|_{\mathbb{S}^{d-1}}$ the space of all polynomials in Π_n^d restricted on $\mathbb{S}^{d-1}$. In the following we shall write Π_n^d for $\Pi_n^d(\mathbb{S}^{d-1})$ whenever it causes no confusion. The quantity of best