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Favard interpolation from subsets of a rectangular lattice

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Abstract

This is a study of Favard interpolation – in which the nth derivatives of the interpolant are bounded above by a constant times the nth divided differences of the data – in the case where the data are given on some subset of a rectangular lattice in $\mathbb{R}^k$. In some instances, depending on the geometry of this subset, we construct a Favard interpolant, and in other instances, we prove that none exists.

Keywords: Interpolation; Divided differences; Multivariate; Favard

1. Introduction

If f is a real-valued function defined either on an infinite set of real numbers

$$\mathbf{m} = \{\cdots < m_{-1} < m_0 < m_1 < \cdots\}$$

or on a finite set

$$\mathbf{m} = \{m_0 < m_1 < \cdots < m_d\},$$

then Rolle's Theorem implies that any smooth extension Ff of f must satisfy

$$\|D^n Ff\|_{L_\infty} \geq n!\|[m_i, \ldots, m_{i+n}]f\|_{\ell_\infty}.$$

Here, $[m_i, \ldots, m_{i+n}]f$ is the nth order divided difference of f formed from its values at $m_i, \ldots, m_{i+n}$, the L_∞ norm is over the interval $(\inf \mathbf{m}, \sup \mathbf{m})$, and the ℓ_∞ norm is taken over

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the collection of all allowable i (depending on the cardinality of $\mathbf{m}$). If the right side is finite, is there an extension Ff satisfying

$$\|D^n Ff\|_{L_\infty} \le C(n)\|[m_i, \ldots, m_{i+n}]f\|_{\ell_\infty} \tag{1.1}$$

for some quantity $C(n)$ depending only on n?

Favard [1] answered this question in the affirmative when he showed that, for every such $\mathbf{m}$, f, and natural number n, not only does such a smooth function Ff exist, but it depends linearly and locally on f in such a way that, even if the divided differences of f are unbounded, $|D^n Ff|$ is locally no larger than the numbers

$$|[m_i, \ldots, m_{i+n}]f|$$

times some $C(n)$. (See [5] for a proof in English.)

(In fact, it has been proven [6] that Favard's interpolant satisfies both (1.1) and

$$\|D^{n-1} Ff\|_{L_\infty} \le C(n)\|[m_i, \ldots, m_{i+n-1}]f\|_{\ell_\infty}$$

simultaneously, and that no interpolant can bound more derivatives by the corresponding divided differences on general $\mathbf{m}$.)

This is the author's fourth paper resulting from investigations into Favard's theorem and analogous theorems for functions f given on a discrete set of points in $\mathbb{R}^k$. In any such theorem, some derivative or collection of derivatives would have to take the place of the nth derivative of the function Ff and some differences would have to serve as the "multivariate" divided differences of order n of f. Until we establish more concrete notation later, we will refer loosely to the former as $(Ff)^{(n)}$ and to the latter as $f^{(n)}$. For instance, $(Ff)^{(n)}$ might consist of all mixed partial derivatives of total order n of Ff, and $f^{(n)}$ could be some derivative information gleaned from f by means of linear functionals that annihilate the k-variate polynomials of degree less than n. *Favard's interpolation problem* is to find an extension Ff of f that satisfies

$$|(Ff)^{(n)}| \le C(n,k)|f^{(n)}|. \tag{1.2}$$

This differs from the generalized Whitney Extension Theorem [8, VI.2, Theorem 4], which, among other things, promises that if f and its derivatives of total order $\le n$ are given on a closed set in $\mathbb{R}^k$, then there exists an extension $\mathcal{E}f$ of f to all $\mathbb{R}^k$ for which

$$\max_{h\le n} |(\mathcal{E}f)^{(h)}| \le C(n,k) \max_{h\le n} |f^{(h)}|. \tag{1.3}$$

To use the Whitney theorem in Favard's problem, given only the function f on a discrete set of data points, one would have to assign values to f's derivatives of order $\le n$ at the data points, presumably by means of finite differences. But even then, (1.3) will not guarantee that $\mathcal{E}f$ satisfies the more restrictive inequality (1.2).

Throughout this paper, $f^{(n)}$ will be taken to be the values at f of certain tensor product divided differences. This means that f must be defined on some subset of a *tensor product grid* of points in $\mathbb{R}^k$, that is, the Cartesian product of k sequences of real numbers

$$\mathbf{m}_i = \left(\cdots < m_{i,-1} < m_{i,0} < m_{i,1} < \cdots\right) \quad (i = 1, \ldots, k).$$

In case each sequence $\mathbf{m}_i$ has constant step size Δ_i, in which case the Cartesian product is a rectangular *lattice*, a simple construction has been proven [4] to produce an interpolant Ff that