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Stechkin's problem for differential operators and functionals of first and second orders

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Abstract

In this paper, we present the solution to Stechkin's problem for differential operators and functionals of first and second orders on the class of functions that are defined on a finite interval and have bounded third derivative. Two related problems are discussed: (1) finding sharp constants in the Landau–Kolmogorov inequalities, and (2) optimal recovery of an operator with the help of a set of linear operators (or arbitrary single-valued mappings) on elements of some set that are given with an error.

Keywords: Unbounded operator; Landau–Kolmogorov inequality; Sharp constants; Stechkin's problem; Extremal problems; Best approximation

1. Introduction. Statement of the problems

The problem of best approximation of an unbounded operator by linear bounded ones was first formulated by S.B. Stechkin in 1967 [36]. The question emerged as part of a method to find exact constants in inequalities for intermediate derivatives of univariate and multivariate functions (see [35]). However, beyond Stechkin's problem being interesting on its own, it is

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also closely connected to other extremal problems of Approximation Theory. Among them are problems of finding best numerical formulas for differentiation of univariate and multivariate functions and the problem of best approximation of classes of functions by some other classes of functions.

At present, there are many solutions of Stechkin's problem for shift-invariant operators on classes of functions defined on the real line. However, much fewer results exist in the case when the domain of the functions is either half-line or a finite interval. The difficulties in these situations arise due to a "perturbation" effect related to the boundary of the domain (problem is no longer invariant with respect to shifts).

In this paper, we present the solution to Stechkin's problem (and a series of related problems) for differential operators and functionals of first and second orders on the class of functions, defined on a finite interval that have bounded third derivative. Let us turn now to necessary notations and statements of the main problems and results.

1.1. Stechkin's problem: statement and history

Let the domain G be the real line $\mathbb{R} = (-\infty, \infty)$, the half-line $\mathbb{R}_+ := [0, +\infty)$, or the interval $\mathbb{I} := [0, 1]$. Let also $L_\infty(G)$ be the space of essentially bounded functions $f : G \to \mathbb{R}$ endowed with the standard norm

$$\|f\|_\infty := \operatorname{ess\,sup} \{|f(x)| : x \in G\}.$$

For every $r \in \mathbb{N}$, by $L^r_\infty(G)$ we shall denote the set of functions $f \in L_\infty(G)$ that have derivative of order $(r-1)$ locally absolutely continuous on G. We set

$$W^r_\infty(G) := \left\{ f \in L^r_\infty(G) : \left\| f^{(r)} \right\|_\infty \le 1 \right\}.$$

Next we fix $k \in \mathbb{N}$, $1 \le k \le r-1$, and a point $t \in G$. In what follows by T we shall denote

1. either differentiation operator D^k of order k;
2. or differentiation functional D^k_t of order k at a point $t \in G$.

Now let us turn to the rigorous statement of Stechkin's problem about best approximation of (unbounded) operator T by linear bounded ones on the class $W^r_\infty(G)$.

Let $Y = L_\infty(G)$ in the case $T = D^k$, and let $Y = \mathbb{R}$ in the case $T = D^k_t$. For $N > 0$ by $\mathcal{L}(N; Y)$ we shall denote the space of all linear operators $F : L_\infty(G) \to Y$ with norm bounded by N. Following [36], for an arbitrary operator $F \in \mathcal{L}(N; Y)$ we introduce the following notation

$$U\left(T, F; W^r_\infty(G)\right) := \sup_{f \in W^r_\infty(G)} \|Tf - Ff\|_Y$$

for the best approximation of operator T with the help of operator F. The next problem is referred to as Stechkin's problem.

Problem 1 (*Stechkin's Problem*)**.** For all $N > 0$ find the best approximation of the operator T by linear bounded operators on the class $W^r_\infty(G)$, i.e.

$$E_N\left(T; W^r_\infty(G)\right) := \inf_{F \in \mathcal{L}(N;Y)} U\left(T, F; W^r_\infty(G)\right), \tag{1}$$

as well as extremal operators $F^* \in \mathcal{L}(N; Y)$ (if any exists), that realize infimum in (1).