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# Orthogonal polynomials of the ℝ-linear generalized minimal residual method

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**Abstract**

The speed of convergence of the ℝ-linear GMRES method is bounded in terms of a polynomial approximation problem on a finite subset of the spectrum. This result resembles the classical GMRES convergence estimate except that the matrix involved is assumed to be condiagonalizable. The bounds obtained are applicable to the CSYM method, in which case they are sharp. Then a new three term recurrence for generating a family of orthogonal polynomials is shown to exist, yielding a natural link with complex symmetric Jacobi matrices. This shows that a mathematical framework analogous to the one appearing with the Hermitian Lanczos method exists in the complex symmetric case. The probability of being condiagonalizable is estimated with random matrices.


*Keywords:* ℝ-linear GMRES; Condiagonalizable; Three term recurrence; Orthogonal polynomial; Jacobi matrix; Spectrum; Polynomial approximation; CSYM; Random matrix

## 1. Introduction

Suggested in [9], there exists an ℝ-linear GMRES (generalized minimal residual) method for solving a large real linear system of equations of the form

$$\kappa z + M_{\#}\overline{z} = b \tag{1.1}$$

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for $\kappa \in \mathbb{C}$, $M_{\#} \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. Systems of this type appear regularly in applications. This is manifested by the complex symmetric case which corresponds to $\kappa = 0$ and ${M_{\#}}^T = M_{\#}$. (For its importance in applications, such as the numerical solution of the complex Helmholtz equation, see [11].) Then the $\mathbb{R}$-linear GMRES method reduces to the CSYM method [5]. We have $\kappa \neq 0$, e.g., in an approach to solve the electrical conductivity problem [2] which requires solving an $\mathbb{R}$-linear Beltrami equation [19]. For a wealth of information regarding real linearity, see [20,7]. Although the $\mathbb{R}$-linear GMRES method is a natural scheme, its properties are not well understood. Assuming $M_{\#}$ to be condiagonalizable, in this paper a polynomial approximation problem on the plane is introduced for assessing its speed of convergence. In the complex symmetric case a new three term recurrence for generating orthogonal polynomials arises, leading to a natural link with complex symmetric Jacobi matrices.

The bounds obtained are intriguing by the fact that they show that the convergence depends on the spectrum of the real linear operator involved. So far it has not been clear what is the significance of the spectrum in general and for iterative methods in particular [16,9]. Here it is shown to play a role similar to what the spectrum does in the classical GMRES bounds [28]. A striking difference is that the bounds reveal a strong dependence of the speed of convergence on the vector.

Moreover, with any natural Krylov subspace method there exists a connection between the iteration and orthogonal functions. As a rule, these are associated with normality. The Hermitian Lanczos method is related with a three term recurrence for generating orthogonal polynomials; see [14] and references therein. For unitary matrices the corresponding length of recurrence is five [27]; see also [29]. These are special instances of the general framework for normal matrices[1] described in [17,18]. In this paper an analogous connection is established in the complex symmetric case to orthogonalize monomials

$$1, \lambda, |\lambda|^2, \lambda|\lambda|^2, |\lambda|^4, \lambda|\lambda|^4, \ldots \tag{1.2}$$

with a three term recurrence. This gives rise to polynomials of the form

$$\sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (\alpha_{2k} + \alpha_{2k+1}\lambda)\, |\lambda|^{2k},$$

with $\alpha_k \in \mathbb{C}$ and, for $j$ even $\alpha_{j+1} = 0$, whose union we denote by $\mathcal{P}(r2)$ by the fact that the related family of functions can be viewed to extend radial functions in a natural way. This connection is not entirely unexpected since antilinear operators involving a complex symmetric matrix $M_{\#}$ have been regarded as yielding an analogue of normality [16, p. 250]. As opposed to the Hermitian Lanczos method, the structure is richer now as orthogonality based on a three term recurrence and respective rapid least squares approximation is possible not just on subsets of $\mathbb{R}$. In particular, we characterize those curves which admit the associated Weierstrass-type polynomial approximation result for $\mathcal{P}(r2)$; see Theorem 4.5.

The polynomial space $\mathcal{P}(r2)$ can also be used to analyze the speed of convergence of the $\mathbb{R}$-linear GMRES method in the condiagonalizable case. Unlike diagonalizability in the complex linear case, condiagonalizability is a more intricate structure. Random matrix theory is invoked to assess how likely it is to have a condiagonalizable operator in (1.1). In this manner we end

[1] The length of recurrence depends on what is the least possible degree for an algebraic curve to contain the eigenvalues.