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Nikishin systems are perfect. The case of unbounded and touching supports

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Abstract

K. Mahler introduced the concept of perfect systems in the theory of simultaneous Hermite–Padé approximation of analytic functions. Recently, we proved that Nikishin systems, generated by measures with bounded support and non-intersecting consecutive supports contained on the real line, are perfect. Here, we prove that they are also perfect when the supports of the generating measures are unbounded or touch at one point. As an application, we give a version of the Stieltjes theorem in the context of simultaneous Hermite–Padé approximation.

Keywords: Perfect systems; Nikishin systems; Multiple orthogonal polynomials; Mixed type orthogonal polynomials; Bi-orthogonal polynomials; Stieltjes theorem

1. Introduction

This paper complements [14] where we solved a long standing problem proving that Nikishin systems (generated by measures whose supports are bounded and consecutive supports do not intersect) are perfect. Here, we consider Nikishin systems whose generating measures may have

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unbounded support or consecutive supports touch at one point. A detailed account on the history of the problem is contained in the introduction of [14], so we will go directly to the subject matter.

1.1. Perfect systems

The concept of perfect systems was introduced and developed by Mahler in lectures delivered at the University of Groningen in 1934–35 which were published much later in [22] (see also [6, 20]). This notion plays a central role in the general theory of simultaneous approximation of systems of analytic functions. Systems of exponential functions are perfect and the corresponding properties were used by Hermite in [19] to prove the transcendence of e. This may have inspired Mahler to introduce the general concept and study its properties.

Mahler's general approach to the simultaneous approximation of finite systems of analytic functions may be reformulated in the following terms.

Let $\mathbf{f} = (f_0, \ldots, f_m)$ be a system of formal power expansions at ∞ of the form

$$f_j(z) = \sum_{n=0}^{\infty} \frac{f_{j,n}}{z^n}, \quad j = 0, \ldots, m.$$

Fix a non-zero multi-index $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{Z}_+^{m+1}$, $|\mathbf{n}| = n_0 + \cdots, n_m$. There exist polynomials $a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m}$, not all identically equal to zero, such that

(i) $\deg a_{\mathbf{n},j} \leq n_j - 1$, $j = 0, \ldots, m$ ($\deg a_{\mathbf{n},j} \leq -1$ means that $a_{\mathbf{n},j} \equiv 0$),
(ii) $\sum_{j=0}^{m} a_{\mathbf{n},j}(z) f_j(z) - b_{\mathbf{n}}(z) = \frac{A_{\mathbf{n}}}{z^{|\mathbf{n}|}} + \cdots$,

for some polynomial $b_{\mathbf{n}}$. Analogously, there exists a polynomial $Q_{\mathbf{n}}$, not identically equal to zero, such that

(i) $\deg Q_{\mathbf{n}} \leq |\mathbf{n}|$,
(ii) $Q_{\mathbf{n}}(z) f_j(z) - P_{\mathbf{n},j}(z) = \frac{A_{\mathbf{n},j}}{z^{n_j+1}} + \cdots$, $j = 0, \ldots, m$,

for some polynomials $P_{\mathbf{n},j}$, $j = 0, \ldots, m$.

The right hand of (ii) must be understood as a formal expansion in decreasing powers of z obtained after carrying out arithmetically the operations of the left hand. Certainly, if the left hand is analytic at ∞ the right hand will be convergent in some neighborhood of ∞ and equality is in the usual sense in that neighborhood.

The polynomials $b_{\mathbf{n}}$ and $P_{\mathbf{n},j}$, $j = 0, \ldots, m$, are uniquely determined from (ii) once their partners $a_{\mathbf{n},j}$, $j = 0, \ldots, m$, and $Q_{\mathbf{n}}$ are found. The two constructions are called type I and type II polynomials (approximants) of the system $(f_0, \ldots, f_m)$, respectively. When $m = 0$ both definitions reduce to the well-known Padé approximation in its linear presentation.

In applications (number theory, convergence of simultaneous rational approximation, asymptotic properties of type I and type II polynomials, non-intersecting Brownian motions, and random matrix theory) it is important that the polynomials appearing in the construction have no defect; that is, that they have full degree. For one, this guarantees uniqueness up to a constant factor. Here, the following concept steps in.

Definition 1.1. A multi-index $\mathbf{n} = (n_0, \ldots, n_m)$ is said to be **normal** for the system $\mathbf{f}$ for type I approximation (respectively, for type II) if $\deg a_{\mathbf{n},j} = n_j - 1$, $j = 0, \ldots, m$ (respectively, $\deg Q_{\mathbf{n}} = |\mathbf{n}|$). A system of functions $\mathbf{f}$ is said to be **perfect** if all multi-indices are normal.

It is easy to see that normality implies that all solutions are collinear.