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Minimal cubature rules and polynomial interpolation in two variables

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Dedicated to my friend Péter Vértesi on the occasion of his 70th birthday

Abstract

Minimal cubature rules of degree $4n-1$ for the weight functions

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x,y) = |x+y|^{2\alpha+1}|x-y|^{2\beta+1}((1-x^2)(1-y^2))^{\pm\frac{1}{2}}$$

on $[-1,1]^2$ are constructed explicitly and are shown to be closely related to the Gaussian cubature rules in a domain bounded by two lines and a parabola. Lagrange interpolation polynomials on the nodes of these cubature rules are constructed and their Lebesgue constants are determined.

Keywords: Cubature; Gaussian; Minimal cubature; Polynomial interpolation; Two variables; Lebesgue constant

1. Introduction

Minimal cubature rules have the smallest number of nodes among all cubature rules of the same precision. Let W be a non-negative weight function on a domain $\Omega \subset \mathbb{R}^2$. For a positive integer s, a cubature rule of precision s with respect to W is a finite sum that satisfies

$$\int_\Omega f(x,y)W(x,y)\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{N} \lambda_k f(x_k,y_k), \quad \forall f \in \Pi_s^2, \tag{1.1}$$

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where Π_s^2 denotes the space of polynomials of degree at most s in two variables, and there exists at least one function f^* in Π_{s+1}^2 such that the Eq. (1.1) does not hold.

It is known that the number of nodes N of a cubature rule of degree s necessarily satisfies

$$N \geq \dim \Pi_{n-1}^2 = \frac{n(n+1)}{2}, \qquad s = 2n-1 \quad \text{or} \quad 2n-2 \tag{1.2}$$

(cf. [12,16]). A cubature rule of degree s with N attaining the lower bound in (1.2) is called Gaussian. Unlike quadrature rules in one variable, Gaussian cubature rules rarely exist. At the moment, they are known to exist only in two cases. The first case, discovered in [13], is for a family of weight functions that includes, in particular, $W_{\alpha,\beta,\pm\frac{1}{2}}$ defined by

$$W_{\alpha,\beta,\pm\frac{1}{2}}(u,v) = (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\pm\frac{1}{2}} \tag{1.3}$$

on the domain $\Omega = \{(u,v) : 1+u+v > 0, 1-u+v > 0, u^2 > 4v\}$, bounded by two lines and a parabola. On the other hand, Gaussian cubature rules of degree $2n-1$ do not exist when W is centrally symmetric, that is, when W and its domain Ω are both symmetric with respect to the origin: $(-x,-y) \in \Omega$ whenever $(x,y) \in \Omega$ and $W(-x,-y) = W(x,y)$. For centrally symmetric weight functions and $s = 2n-1$, a stronger lower bound [10] for the number of nodes is given by

$$N \geq \dim \Pi_{n-1}^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{n(n+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor. \tag{1.4}$$

A cubature rule that attains this lower bound is necessarily minimal. There are, however, only a couple of examples for which this lower bound is attained for all n, most notable being the product Chebyshev weight functions on the square.

In the present paper we shall show that the minimal cubature rules of degree $4n-1$ exist for a family of weight functions that includes, in particular,

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x,y) := |x+y|^{2\alpha+1}|x-y|^{2\beta+1}(1-x^2)^{\pm\frac{1}{2}}(1-y^2)^{\pm\frac{1}{2}}, \tag{1.5}$$

on $[-1,1]^2$ and, furthermore, there is a connection between these minimal cubature rules and Gaussian cubature rules associated with the weight function $W_{\alpha,\beta,\pm\frac{1}{2}}$. The weight functions (1.5) include the product Chebyshev weight functions (when $\alpha = \beta = \pm\frac{1}{2}$), for which the minimal cubature rules are known to exist and have been established in several different methods [1,9,11,20]. Our result shows that they can be deduced from the Gaussian cubature rules for $W_{-\frac{1}{2},-\frac{1}{2},\pm\frac{1}{2}}$ on Ω. Giving the fact that so few minimal cubature rules are known explicitly, this connection is rather surprising.

Cubature rules are closely related to interpolation by polynomials. Based on the nodes of a Gaussian cubature rule of degree $2n-1$, there is a unique Lagrange interpolation polynomial of degree $n-1$ which converges to f in L^2 norm as $n \to \infty$ [19]. On the nodes of the minimal cubature rule that attains (1.4), there is a unique Lagrange interpolation polynomial in an appropriate subspace of polynomials [20]. Furthermore, the interpolation polynomials based on the nodes of the minimal cubature rules for the product Chebyshev weight function $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$, studied in [21], has the Lebesgue constant of order $(\log n)^2$ [3], which is the minimal order of projection operators on $[-1,1]^2$ [18]. We shall discuss the Lagrange interpolations based on both