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The third Appell function for one large variable

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Abstract

We consider the Mellin convolution integral representation of the third Appell function given in Erdélyi (1953) [8]. Then, we apply the asymptotic method designed in Wong (1979) [22] and revisited in López (2008) [14] for this kind of integral to derive new asymptotic expansions of the Appell function F_3 for one large variable in terms of hypergeometric functions. The accuracy of the approximations is illustrated with numerical experiments.

Keywords: Third Appell hypergeometric function; Asymptotic expansions; Mellin convolution integrals

1. Introduction

The Appell functions F_1, F_2, F_3 and F_4 were introduced in 1880 by Paul Appell as a generalization of the Gauss hypergeometric function ${}_2F_1$. They are defined by means of double power series and cannot be expressed as a product of two ${}_2F_1$ functions. In particular, the third Appell function is defined by

$$F_3(a, a', b, b', c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n} m! n!} x^m y^n, \quad \max(|x|, |y|) < 1. \tag{1}$$

Appell functions have many applications as solutions of certain ordinary and partial differential equations in several areas of physics: quantum mechanics, transition matrices in atomic and

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molecular physics, mechanical systems and others [5–7,13,21]. There is an extensive literature devoted to the study of these functions. In particular, Erdélyi's book [8] is a classic reference that contains various types of information about these functions: simple and double Mellin–Barnes type integral representations, partial differential equations, transformation and reduction formulas, etc. In recent decades, several authors have derived other interesting results for the Appell functions. Sharma has obtained generating functions of the Appell functions [20]. Some integral representations have been derived by Mittal [15]. Exton has obtained the Laplace transforms of these functions [9]. Some reduction formulas for special values of the variables and contiguous relations for Appell functions have been investigated by Buschman [1,2]. Carlson has investigated quadratic transformations of Appell functions [4] and their role in multiple averages [3]. The relation between the dilogarithm and Appell function F_3 is examined in [18]. Some results on fractional calculus operators involving F_3 are given in [11,19].

On the other hand, the study of the asymptotic behavior of the Appell functions for large values of x and/or y has barely been considered in the literature, as it can be checked in the recently published NIST Digital Library of Mathematical Functions [16, Section 16.5]. In previous papers [10,12], we have analyzed the asymptotic behavior of the Appell functions F_1 and F_2 for large values of their variables. In particular, convergent and asymptotic expansions of $F_1(a, b, c, d; x, y)$ for large values of x and/or y have been derived in [10]. In [12], asymptotic expansions of $F_2(a, b, b', c, c'; x, y)$ have been obtained for large values of x and y with x/y bounded.

In this paper, we continue the asymptotic study started in [10,12]. We investigate the asymptotic behavior of $F_3(a, a', b, b', c; x, y)$ for one large variable (large x and fixed y or *vice-versa*). As in [10,12], we use here the general asymptotic technique designed by Wong in [22] and revisited in [14] for Mellin convolution integrals, and that we briefly resume in the rest of this section. It can be applied to Mellin convolution integrals of the form

$$F(x) := \int_0^\infty h(xt) f(t) dt, \quad x \to 0^+.$$

The technique requires for $f(t)$ and $h(t)$ to be locally integrable on $(0, \infty)$ and they have the following asymptotic behavior.

(H1) $f(t)$ has a power asymptotic expansion at $t \to \infty$,

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{\alpha_k}} + f_n(t), \quad n = 1, 2, 3, \ldots,$$

where α_k is an increasing sequence of real numbers, $a_k \in \mathbb{C}$, $f_n(t) = \mathcal{O}(t^{-\alpha_n})$ as $t \to \infty$ and, as $t \to 0^+$: $f(t) = \mathcal{O}(t^{-\alpha})$, $\alpha \in \mathbb{R}$.

(H2) $h(t)$ has a power asymptotic expansion at $t \to 0^+$,

$$h(t) = \sum_{k=0}^{m-1} b_k t^{\beta_k} + h_m(t), \quad m = 1, 2, 3, \ldots,$$

where β_k is an increasing sequence of real numbers, $b_k \in \mathbb{C}$, $h_m(t) = \mathcal{O}(t^{\beta_m})$ as $t \to 0^+$ and, as $t \to \infty$: $h(t) = \mathcal{O}(t^{-\beta})$, $\beta \in \mathbb{R}$.

The technique also requires, without loss of generality (see [14] for a full explanation), the following relations for the parameters α, β, α_0 and β_0:

(H3) $\alpha - \beta_0 < 1$, $\beta + \alpha_0 > 1$, $\beta + \beta_0 > 0$ and $\alpha < \alpha_0$.