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Best one-sided L_1 approximation to the Heaviside and sign functions

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Abstract

We find the polynomials of the best one-sided approximation to the Heaviside and sign functions. The polynomials are obtained by Hermite interpolation at the zeros of some Jacobi polynomials. Also we give an estimate of the error of approximation and characterize the extremal points of the convex set of the best approximants.

Keywords: One-sided approximation; Jacobi polynomials; Hermite interpolation

1. Introduction

Let $\mathbb{P}_n$ denote the family of all algebraic polynomials of degree not greater than n. For $f \in L_1[-1, 1]$ bounded from below (above) the error of the best one-sided approximation to f from below (above) of degree n is defined by

$$E_n^-(f)_1 = \inf \{\|f - P\|_1 : P \in \mathbb{P}_n,\ P \le f\}$$
$$\left(E_n^+(f)_1 = \inf \{\|f - P\|_1 : P \in \mathbb{P}_n,\ f \le P\}\right).$$

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We will denote by $\mathcal{P}_n^-(f)$ ($\mathcal{P}_n^+(f)$) the set of all polynomials in $\mathbb{P}_n$ such that the above infimum is attained. It is known that these sets are not empty. Note that $P \in \mathcal{P}_n^-(f)$ if and only if $P \le f$ on $[-1, 1]$ and

$$\int_{-1}^{1} Q(x)\,dx \le \int_{-1}^{1} P(x)\,dx, \tag{1}$$

for all $Q \in \mathbb{P}_n$ satisfying $Q \le f$ on $[-1, 1]$.

In this paper we obtain explicit expression for polynomials of best one-sided approximation to the Heaviside function

$$G(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0, \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

The polynomials of best one-sided approximation to the function G and to the sign function

$$\mathrm{S}(x) := \mathrm{sign}\,(x) = \begin{cases} -1, & \text{if } -1 \le x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \le 1 \end{cases}$$

are closely related. In fact, since $\mathrm{S}(x) = 2G(x) - 1$, almost everywhere, it is easy to show that $P \in \mathcal{P}_n^-(G)$ if and only if $2P - 1 \in \mathcal{P}_n^-(\mathrm{S})$.

On the other hand, it is known that, if $f \in L_1[-1, 1]$ is a bounded odd function, then $P(x) \in \mathcal{P}_n^+(f)$ if and only if $-P(-x) \in \mathcal{P}_n^-(f)$. Therefore, in what follows we only consider best one-sided approximation from below.

In Section 3 (see Theorem 1) we characterize the polynomials in $\mathcal{P}_n^-(G)$. We show that these polynomials interpolate G at some points $y_{n,i}$. Thus, in order to obtain a complete description of the polynomials P we should identify the points $y_{n,i}$. In Theorem 2 we prove that these points are just the zeros of some Jacobi polynomials. Finally, in Section 4 we give explicit expressions for the polynomials in $\mathcal{P}_n^-(G)$.

Several problems concerning best one-sided approximation have been studied before. In particular, the asymptotic properties of $E_n^-(G)$ were considered in [3] and methods for obtaining best one-sided approximants were developed in [1,2]. Here we follow ideas similar to the ones used in the two last papers. We refer the reader to the classical monograph of Pinkus [4] for a detailed description of some results related to the existence, characterization and uniqueness of the best one-sided approximation.

Application of the results presented here to the construction of operators for one-sided approximation will appear in a forthcoming paper.

2. Some facts related to Jacobi polynomials

For $\alpha > -1, \beta > -1$ and $r \in \mathbb{N}$, the Jacobi polynomial $P_r^{(\alpha,\beta)}$ is the unique polynomial of degree r which satisfies

$$\int_{-1}^{1} Q_{r-1}(x) P_r^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = 0, \quad \text{for all } Q_{r-1} \in \mathbb{P}_{r-1}, \tag{2}$$

and

$$P_r^{(\alpha,\beta)}(1) = \binom{r+\alpha}{r}.$$