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A generalization of the Riesz–Fischer theorem and linear summability methods[☆]

B. Brive, C. Finet[*], G.E. Tkebuchava[1]

[a] *Université de Mons, 8, Avenue du Champ de Mars, BE-7000 Mons, Belgium*

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Abstract

We extend the classical Riesz–Fischer theorem to biorthogonal systems of functions in Orlicz spaces: from a given double series (not necessarily convergent but satisfying a growth condition) we construct a function (in a given Orlicz space) by a linear summation method, and recover the original double series via the coefficients of the expansion of this function with respect to the biorthogonal system. We give sufficient conditions for the regularity of some linear summation methods for double series. We are inspired by a result of Fomin who extended the Riesz–Fischer theorem to L^p spaces.

Keywords: Linear summability methods; Expansions of functions; Biorthogonal systems; Orlicz spaces

1. Introduction

In this work, we extend the classical Riesz–Fischer theorem to biorthogonal systems of functions in Orlicz spaces. Our method can be used to obtain some other results for biorthogonal systems of functions in more general functional spaces. Our construction appears as a particular case of a linear summability method for (possibly) non-convergent double series, for which we

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* Corresponding author.
E-mail addresses: bruno.brive@umons.ac.be (B. Brive), catherine.finet@umons.ac.be (C. Finet).

1 This work was initiated by C. Finet and G.E. Tkebuchava, who died in the meantime, and continued by C. Finet and B. Brive. G.E. Tkebuchava was a member of Tbilisi State University.

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give and prove a more general statement. This is both a classical and an active field of research; although it is not possible to give an extensive bibliography on the subject, we point out the works of Abilov and Kerimov [1], Getsadze [12,11], Kantawala [14], Andrienko and Kovalenko [5,4], Andrienko [3,2], Móricz [18], Móricz and Tandori [20], Móricz and Szalay [19], Chen [6], Rhoades [23], Szalay [25,24], Patel [21]. Summability methods are a powerful tool in Fourier analysis and have applications for example in numerical analysis and the study of mathematical physics equations, see the work of Cheong [7,8].

Let I be a bounded interval in $\mathbf{R}$. If f and g are two Lebesgue integrable real-valued functions on I, we write

$$(f|g) = \int_I f(x)g(x)\,dx.$$

Let $(\varphi_n)_{n\in\mathbf{N}}$ be an orthonormal sequence in $L^2(I)$, that is $(\varphi_n|\varphi_m) = \delta_{n,m}$ (Kronecker symbol) for n and m in $\mathbf{N}$. The classical Riesz–Fischer theorem asserts that given a sequence of real numbers $(c_n)_{n\in\mathbf{N}}$ in $\ell^2(\mathbf{N})$, there exists a function f in the space $L^2(I)$ such that $c_n = (f|\varphi_n)$ for all $n \in \mathbf{N}$. In other words, the numbers c_n are the coefficients of the expansion of the function f in the orthonormal system $(\varphi_n)_{n\in\mathbf{N}}$.

This theorem may be extended in several directions. We can ask for its validity in other spaces of functions than $L^2(I)$, or for double orthonormal systems rather than simple ones.

In [10], Fomin extended the classical one-dimensional Riesz–Fischer theorem to the L^p spaces, $1 \leqslant p < \infty$. He observed that given a sequence of real numbers $(c_n)_{n\in\mathbf{N}}$, the condition

$$\sum_{n\in\mathbf{N}} |c_n|^2 < \infty \tag{1.1}$$

is equivalent to the condition:

> there exists an increasing sequence of positive numbers $(v_n)_{n\in\mathbf{N}}$ with $v_n \to \infty$ as $n \to \infty$, such that
>
> $$\sum_{k=0}^{\infty} \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right) \int_I \left|\sum_{m=0}^{k} c_m v_m \varphi_m(x)\,dx\right|^2 < \infty.$$

This led him to the following analogue of the Riesz–Fischer theorem in $L^p(I)$ ([10, Theorem 1]):

Theorem (*Fomin*)**.** *Let $1 \leqslant p < \infty$, $(\varphi_n)_{n\in\mathbf{N}}$ be an orthonormal system with $\varphi_n \in L^q(I)$, where q is the conjugate exponent to p ($\frac{1}{p} + \frac{1}{q} = 1$) and $(c_n)_{n\in\mathbf{N}}$ be a sequence of real numbers. If for some increasing sequence of positive numbers $(v_n)_{n\in\mathbf{N}}$ with $v_n \to \infty$ as $n \to \infty$ we have*

$$\sum_{k=0}^{\infty} \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right) \int_I \left|\sum_{m=0}^{k} c_m v_m \varphi_m(x)\,dx\right|^p < \infty,$$

then there exists a function $f \in L^p(I)$ such that $c_n = (f|\varphi_n)$ for all $n \in \mathbf{N}$.

The previous theorem was extended to Orlicz classes by Mazhar [17].

Our aim is to extend the Riesz–Fischer Theorem to double orthonormal systems in Orlicz spaces.

To state our result we need to recall some definitions and introduce some notations.