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Notes

Notes on a classic theorem of Erdős and Grünwald

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Dedicated to the memory of Géza Grünwald (1910–1942) on his hundredth anniversary

Abstract

We establish a companion result to a classic theorem of Erdős and Grünwald on the maximum of the fundamental functions of Lagrange interpolation based on the Chebyshev nodes.

Keywords: Lagrange interpolation; Chebyshev polynomial; Fundamental function of interpolation

Let $T_n(x) := \cos(n \arccos x)$ be the Chebyshev polynomial of degree n, with roots

$$x_{k,n} := \cos t_{k,n}, \quad t_{k,n}: = \frac{2k-1}{2n}\pi, \ k = 1, \dots, n.$$

Further let

$$\ell_{k,n}(x) := \frac{(-1)^{k+1} \cos nt \sin t_{k,n}}{n(\cos t - \cos t_{k,n})}, \quad x = \cos t, \ k = 1, \dots, n \tag{1}$$

be the fundamental polynomials of Lagrange interpolation based on the Chebyshev nodes. Erdős and Grünwald proved the following theorem.

Theorem A (*Erdős–Grünwald [1]*). *We have*

$$|\ell_{k,n}(x)| < \frac{4}{\pi}, \quad |x| \le 1, \ 1 \le k \le n, \ n = 1, 2, \dots. \tag{2}$$

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Moreover,

$$\lim_{n\to\infty} \ell_{1,n}(1) = \lim_{n\to\infty} \ell_{n,n}(-1) = \frac{4}{\pi}. \tag{3}$$

The limit relation (3) implies that (2) is sharp. With the notation

$$M_n(x) := \max_{1\le k\le n} \ell_{k,n}(x), \quad |x| \le 1$$

it follows from Theorem A that

$$\lim_{n\to\infty} \max_{|x|\le 1} M_n(x) = \frac{4}{\pi}.$$

Once this value is found, it is natural to ask for the behavior of the minimum of this function. In this connection we prove the following theorem.

Theorem B. *We have*

$$\lim_{n\to\infty} \min_{|x|\le 1} M_n(x) = \frac{2}{\pi} \cos \frac{2-\sqrt{3}}{2}\pi = 0.580\ldots. \tag{4}$$

Proof. First we prove a lower estimate for $M_n(x)$. By symmetry, we may assume that $0 \le x \le 1$. We distinguish three cases.

Case 1: $x_{1,n} \le x \le 1$. Then evidently $M_n(x) \ge \ell_{1,n}(x) \ge 1$, which is bigger than the right-hand side of (4).

Case 2: $x_{4,n} \le x \le x_{1,n}$. Let $x_{k+1,n} \le x \le x_{k,n}$, $1 \le k \le 3$. There exists a unique $x_{k+1,n} < y_{k,n} < x_{k,n}$ such that $\ell_{k+1,n}(y_{k,n}) = \ell_{k,n}(y_{k,n})$. Using (1), an easy calculation yields

$$y_{k,n} = \frac{\sin t_{k+1,n} \cos t_{k,n} + \cos t_{k+1,n} \sin t_{k,n}}{\sin t_{k,n} + \sin t_{k+1,n}} = \frac{\cos \frac{k\pi}{n}}{\cos \frac{\pi}{2n}}.$$

Now if $\ell_{k,n}(y_{k,n}) \ge 1$, then

$$M_n(x) \ge \ell_{k,n}(x) \ge 1, \qquad x_{k+1,n} \le x \le x_{k,n},$$

and we are done just like in Case 1. If $\ell_{k\,,n}(y_{k,n}) < 1$, then

$$\begin{aligned} M_n(x) &\ge \max(\ell_{k+1,n}(x), \ell_{k,n}(x)) \ge \ell_{k,n}(y_{k,n}) = \frac{|T_n(y_{k,n})| \sin t_{k,n}}{n(\cos t_{k,n} - y_{k,n})} \\ &= \frac{|T_n(y_{k,n})|}{n} \cdot \frac{\sin \frac{2k-1}{2n}\pi \cos \frac{\pi}{2n}}{\cos \frac{\pi}{2n} \cos \frac{2k-1}{2n}\pi - \cos \frac{k\pi}{n}} = \frac{2|T_n(y_{k,n})|}{n} \cdot \frac{\sin \frac{2k-1}{2n}\pi \cos \frac{\pi}{2n}}{\cos \frac{k-1}{n}\pi - \cos \frac{k\pi}{n}} \\ &= \frac{|T_n(y_{k,n})|}{n} \cdot \frac{\sin \frac{2k-1}{2n}\pi \cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n} \sin \frac{2k-1}{2n}\pi} = \frac{|T_n(y_{k,n})|}{n} \cot \frac{\pi}{2n}. \end{aligned} \tag{5}$$

Here, by a well-known representation for Chebyshev polynomials,

$$\begin{aligned} T_n(y_{k,n}) &= T_n\left(\frac{\cos \frac{k\pi}{n}}{\cos \frac{\pi}{2n}}\right) \\ &= \frac{1}{2\cos^n \frac{\pi}{2n}} \left[\left(\cos \frac{k\pi}{n} + \mathrm{i}\sqrt{\cos^2 \frac{\pi}{2n} - \cos^2 \frac{k\pi}{n}} \right)^n \right. \end{aligned}$$