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ScienceDirect
Journal of Approximation Theory 162 (2010) 678–686
Journal of Approximation Theory
www.elsevier.com/locate/jat

On inverse problem leading to second-order linear functionals

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Received 12 February 2009; received in revised form 12 June 2009; accepted 27 July 2009
Available online 7 August 2009

Communicated by Sergey Krushchev

Abstract

A linear functional $\mathfrak{L}$ is said to be positive-definite if and only if $\langle \mathfrak{L}, p^2 \rangle > 0$, for all non-zero polynomials with real coefficients p. In this paper, we provide a new construction process of a positive-definite linear functional from positive-definite linear functional data. Indeed, for any non-zero real ϵ and any positive-definite linear functional $\mathfrak{L}$, we show that the linear functional $\mathfrak{L}_\epsilon$ satisfying $\mathfrak{L}_\epsilon - \epsilon \mathfrak{L}'_\epsilon = \mathfrak{L}$ is also positive-definite. This process allows us to construct a second-order positive-definite linear functional from a semiclassical positive-definite linear functional. Finally, we give an illustrative example.

Keywords: Orthogonal polynomials; Recurrence relations; Semiclassical linear functionals; Second-order linear functionals; Laguerre polynomials; Integral representations

1. Introduction and preliminary results

Let $\mathbb{P}$ be the vector space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space. We denote by $\langle \mathfrak{L}, p \rangle$ the action of $\mathfrak{L} \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(\mathfrak{L})_n := \langle \mathfrak{L}, x^n \rangle,\ n \geq 0$, the sequence of moments of $\mathfrak{L}$ with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$. Let us define the following operations in $\mathbb{P}'$. For any linear functional $\mathfrak{L}$, any polynomial q, and any $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}^2$, let $D\mathfrak{L} = \mathfrak{L}',\ q\mathfrak{L},\ \tau_b\mathfrak{L}, h_a\mathfrak{L}$, and $(x - c)^{-1}\mathfrak{L}$ be the linear functionals defined by duality

$$\langle \mathfrak{L}', f \rangle := -\langle \mathfrak{L}, f' \rangle,$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2009.07.007

$$\langle q\mathfrak{L}, f\rangle := \langle \mathfrak{L}, qf\rangle,$$
$$\langle \tau_{-b}\mathfrak{L},\ f\rangle := \langle \mathfrak{L}, \tau_b f\rangle = \langle \mathfrak{L}, f(x-b)\rangle,$$
$$\langle h_a\mathfrak{L},\ f\rangle := \langle \mathfrak{L}, h_a f\rangle = \langle \mathfrak{L}, f(ax)\rangle,$$
$$\langle (x-c)^{-1}\mathfrak{L}, f\rangle := \langle \mathfrak{L}, \theta_c f\rangle = \left\langle \mathfrak{L}, \frac{f(x)-f(c)}{x-c}\right\rangle, \quad f \in \mathbb{P}.$$

For any $c \in \mathbb{C}$, we have

$$(x-c)^{-1}(x-c)\mathfrak{L} = \mathfrak{L} - (\mathfrak{L})_0\delta_c, \tag{1.1}$$

where δ_c is the Dirac mass, defined by $\langle \delta_c, p\rangle = p(c),\ p \in \mathbb{P}$.

Notice that if $\mathfrak{L} \in \mathbb{P}'$ is such that $\mathfrak{L}' = 0$, then $\mathfrak{L} = 0$.

A linear functional $\mathfrak{L}$ is said to be quasi-definite (regular) if we can associate with it a monic polynomial sequence $\{B_n\}_{n\geq 0}$, $\deg B_n = n,\ n \geq 0$, such that $\langle \mathfrak{L}, B_n B_m\rangle = r_n\delta_{n,m},\ m,\ n \geq 0$, with $r_n \neq 0,\ n \geq 0$. In such a case, $\{B_n\}_{n\geq 0}$ is said to be the monic orthogonal polynomial sequence (MOPS) with respect to $\mathfrak{L}$.

When $r_n > 0$, for every non-negative integer $n \geq 0$, $\mathfrak{L}$ is said to be positive-definite. Notice that $\mathfrak{L}$ is positive-definite if and only if $\langle \mathfrak{L}, p^2\rangle > 0$, for every non-zero real polynomial p (see [2]).

If $\mathfrak{L}$ is quasi-definite and A is a polynomial such that $A\mathfrak{L} = 0$, then $A = 0$.

Definition 1 (*[7]*). $\mathfrak{L}$ is said to be a second-order linear functional if it is quasi-definite and satisfies

$$\big((\phi\mathfrak{L})' + \psi\mathfrak{L}\big)' + \chi\mathfrak{L} = 0, \tag{1.2}$$

where ϕ, ψ, χ, are polynomials and ϕ monic.

Furthermore, the MOPS $\{B_n\}_{n\geq 0}$ with respect to $\mathfrak{L}$ is also said to be of second order.

The family of second-order linear functionals contains the semiclassical ones (see [7]), when $\chi(x) \equiv 0$ and $\deg \psi \geq 1$, i.e., if the quasi-definite linear functional $\mathfrak{L}$ satisfies

$$(\phi\mathfrak{L})' + \psi\mathfrak{L} = 0. \tag{1.3}$$

In such a case, the class of $\mathfrak{L}$ is the minimum value of $\max(\deg(\phi) - 2,\ \deg(\psi) - 1)$, among all possible pairs (ϕ, ψ) of polynomials satisfying (1.3). The pair (ϕ, ψ) giving the class $s \geq 0$ is unique. When $s = 0$, the linear functional $\mathfrak{L}$ is said to be classical (Hermite, Laguerre, Bessel, Jacobi) and $\deg \phi \leq 2$, $\deg \psi = 1$.

Any shift leaves invariant the semiclassical character. Indeed, the shifted linear functional $\widetilde{\mathfrak{L}} = (h_{a^{-1}} \circ \tau_{-b})\mathfrak{L}$ fulfills the equation $(\widetilde{\phi}\widetilde{\mathfrak{L}})' + \widetilde{\psi}\widetilde{\mathfrak{L}} = 0$, where $\widetilde{\phi}(x) = a^{-\deg\phi}\phi(ax+b)$ and $\widetilde{\psi}(x) = a^{1-\deg\phi}\psi(ax+b)$. For more details, the reader is referred to [7].

When a second-order linear functional $\mathfrak{L}$ is not semiclassical, it is said to be a strict second-order one linear functional.

The theory of the strict second-order linear functionals has not been developed sufficiently. In the literature, we find very few examples of such a kind of linear functionals (see [1,4,7]). We can mention the linear functional u given by the following integral representation [7, p. 126]:

$$\langle u, f\rangle = \frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty} x^{\frac{\alpha-1}{2}} e^{-\sqrt{x}} f(x)\mathrm{d}x, \quad f \in \mathbb{P},$$