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Global asymptotics of orthogonal polynomials associated with $|x|^{2\alpha}e^{-Q(x)}$

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Abstract

In this paper, we consider the asymptotics of polynomials orthogonal with respect to the weight function $w(x) = |x|^{2\alpha}e^{-Q(x)}, \alpha > -\frac{1}{2}$, where $Q(x) = \sum_{k=0}^{2m} q_k x^k, q_{2m} > 0, m > 0$ is a polynomial of degree $2m$. Globally uniform asymptotic expansions are obtained for z in four regions. These regions together cover the whole complex z-plane. Due to the singularity of $|x|^{2\alpha}$, the expansion in the region containing the origin involves Bessel functions. We also study the asymptotic behavior of the leading coefficients and the recurrence coefficients of these polynomials. Our approach is based on a modified version of the steepest descent method for Riemann–Hilbert problems introduced by Deift and Zhou [P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems, Asymptotics for the mKdV equation, Ann. of Math. 137 (1993) 295–368].

Keywords: Orthogonal polynomials; Global asymptotics; Riemann–Hilbert problems; Airy functions; Bessel functions

1. Introduction

In this paper, we consider the weight function defined on the real line by

$$w(x) = |x|^{2\alpha}e^{-Q(x)}, \quad \alpha > -\frac{1}{2}, \tag{1.1}$$

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where

$$Q(x) = \sum_{k=0}^{2m} q_k x^k, \quad q_{2m} > 0, m > 0 \tag{1.2}$$

is a polynomial of degree $2m$. Denote by

$$\pi_n(x) = \pi_n(x, Q) = x^n + \cdots \tag{1.3}$$

the monic polynomials orthogonal with respect to $w(x)$, and let $p_n(x) = \gamma_n \pi_n(x)$ be the normalized orthogonal polynomials, i.e.,

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) |x|^{2\alpha} \mathrm{e}^{-Q(x)} \mathrm{d}x = \delta_{n,m} \quad n, m \in \mathbb{N}. \tag{1.4}$$

The functions $p_n(x)$ satisfy the three-term recurrence relation

$$x p_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x), \tag{1.5}$$

where $b_n = \gamma_n / \gamma_{n+1}$.

The main purpose of this paper is to provide global asymptotic expansions for $\pi_n(z)$ as well as the asymptotic formulas for the leading coefficients γ_n of the polynomials $p_n(z)$ and the recurrence coefficients a_n, b_{n-1} given in (1.5) as $n \to \infty$. In the literature, various special cases or modification of our problem have been investigated. First, in [10] Magnus used Freud difference equation to study the asymptotic behavior of b_{n-1} as $n \to \infty$ when $Q(x) = x^{2m}$. More precisely, he showed that the quotient $b_{n-1}/n^{1/2m}$ tends to a constant depending only on m as $n \to \infty$, which is an important special case of Freud's conjecture. An extension of Magnus' result to Q being an even polynomial of fixed degree with nonnegative coefficients can be found in [3]. We also mention [9], in which the authors obtained bounds on the orthonormal polynomials by requiring $Q(x)$ to be an even, positive and twice continuously differentiable function defined on the real line. In particular, if $\alpha = 0$ in (1.1), our case reduces to the exponential weights considered in [6], and we refer to references therein for this very special case.

To state the asymptotic behavior of the leading coefficients γ_n and the recurrence coefficients a_n, b_{n-1}, we need to introduce some notations. Denote by $N = n + \alpha$ and set

$$Q_N(x) := \frac{Q(N^{1/2m} x)}{N} = \sum_{k=0}^{2m} N^{(k-2m)/2m} q_k x^k. \tag{1.6}$$

Let μ_N be the equilibrium measure associated with the external field $Q_N(x)$, see, e.g., [13]. This measure is defined as the unique one which minimizes the energy functional

$$I(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|z-t|} \mathrm{d}\mu(z) \mathrm{d}\mu(t) + \int_{\mathbb{R}} Q_N(x) \mathrm{d}\mu(x) \tag{1.7}$$

among all the probability measures μ over the real line. It turns out that for sufficiently large N, μ_N is supported on an interval $[\alpha_N, \beta_N]$ with $\alpha_N < 0 < \beta_N$ and its probability density function is given by

$$\mu_N(x) = \frac{1}{2\pi} \sqrt{(\beta_N - x)(x - \alpha_N)} h_N(x), \tag{1.8}$$