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ℓ_1-summability of higher-dimensional Fourier series

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Abstract

It is proved that the maximal operator of the ℓ_1-Fejér means of a d-dimensional Fourier series is bounded from the periodic Hardy space $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for all $d/(d+1) < p \le \infty$ and, consequently, is of weak type (1, 1). As a consequence we obtain that the ℓ_1-Fejér means of a function $f \in L_1(\mathbb{T}^d)$ converge a.e. to f. Moreover, we prove that the ℓ_1-Fejér means are uniformly bounded on the spaces $H_p(\mathbb{T}^d)$ and so they converge in norm $(d/(d+1) < p < \infty)$. Similar results are shown for conjugate functions and for a general summability method, called θ-summability. Some special cases of the ℓ_1–θ-summation are considered, such as the Weierstrass, Picard, Bessel, Fejér, de la Vallée Poussin, Rogosinski and Riesz summations.

Keywords: Hardy spaces; p-atom; Interpolation; Fourier series; ℓ_1–θ-summation

1. Introduction

It is known that Carleson's theorem holds for higher dimensions. More exactly,

$$s_k f(x) := \sum_{j\in\mathbb{Z}^d, |j|\le k} \hat{f}(j) e^{\imath j\cdot x} \to f(x) \quad \text{for a.e. } x \in \mathbb{T}^d \text{ as } k \to \infty$$

if $f \in L_p(\mathbb{T}^d)$ $(1 < p < \infty)$, where $|\cdot| = \|\cdot\|_1$ or $|\cdot| = \|\cdot\|_\infty$ (see [5,8,11]). This is false for $p = 1$. However, in the one-dimensional case the Fejér, Riesz, Weierstrass, Abel, etc. summability means $\sigma_n f$ of f converge to f almost everywhere if $f \in L_1(\mathbb{T})$ (see [25,4,16] or [19]).

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In this paper we generalize the summation results to dimensions $d \geq 2$. We consider the ℓ_1-Fejér means

$$\sigma_n f(x) := \sum_{j \in \mathbb{Z}^d, |j| \leq n} \left(1 - \frac{|j|}{n}\right) \hat{f}(j) e^{\imath j \cdot x}.$$

For $|\cdot|$ denoting the $\|\cdot\|_\infty$ norm, the summation was investigated in [12,24,20]; for the $\|\cdot\|_2$ norm, the summation was investigated in [16,6,11]. In this paper we consider the triangular or ℓ_1-summability, i.e., where $|\cdot| = \|\cdot\|_1$ (see [1,2,23] and more recently [18]). Because of the complexity of the kernel function, this case is rarely investigated in the literature. Since the kernel functions are very different for each norm, we need essentially different ideas. Berens et al. [1] proved for the Fourier transform that $\sigma_T f \to f$ in $L_p(\mathbb{R}^d)$ norm and a.e. as $T \to \infty$, when $f \in L_p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ (for the norm convergence see also [14]).

We will give a sharp estimation for the Fejér kernel function and for its derivative. Using this we generalize the results just mentioned. We prove that $\sigma_n f \to f$ in B-norm, where B is a homogeneous Banach space, which includes the norm convergence in $L_p(\mathbb{T}^d)$ $(1 \leq p < \infty)$ and in $C(\mathbb{T}^d)$. Next we obtain that the maximal operator σ_* is bounded from the Hardy space $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for all $d/(d+1) < p \leq \infty$. This implies by interpolation that σ_* is of weak type (1, 1). As a consequence we get the a.e. convergence of $\sigma_n f$ to f, whenever $f \in L_1(\mathbb{T}^d)$. Moreover, we prove that the Fejér means σ_n are uniformly bounded on the spaces $H_p(\mathbb{T}^d)$ and so they converge in norm $(d/(d+1) < p < \infty)$.

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes all summations mentioned above, is also considered. Similar results are shown for the θ-summation and for conjugate functions and means.

2. Hardy spaces

Let us fix $d \geq 2$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself, d times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_p := \left(\sum_{k=1}^d |x_k|^p\right)^{1/p}, \qquad |x| := \|x\|_1.$$

We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{T}^d} |f|^p \, d\lambda)^{1/p}$, $(0 < p \leq \infty)$, where $\mathbb{T} := [-\pi, \pi]$ is the torus and λ the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. The *weak* L_p space, $L_{p,\infty}(\mathbb{T}^d)$ $(0 < p < \infty)$, consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}(\mathbb{T}^d)$ is a quasi-normed space (see [3]). It is easy to see that for each $0 < p < \infty$,

$$L_p(\mathbb{T}^d) \subset L_{p,\infty}(\mathbb{T}^d) \quad \text{and} \quad \|\cdot\|_{L_{p,\infty}} \leq \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^d)$.

The *Hardy space* $H_p(\mathbb{T}^d)$ $(0 < p \leq \infty)$ consists of all distributions f for which

$$\|f\|_{H_p} := \|\sup_{0 < t} |f * P_t^d|\|_p < \infty,$$