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Computing the Hessenberg matrix associated with a self-similar measure

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Dedicated with affection to Guillermo López Lagomasino on the occasion of his 60th birthday

Abstract

We introduce in this paper a method to calculate the Hessenberg matrix of a sum of measures from the Hessenberg matrices of the component measures. Our method extends the spectral techniques used by G. Mantica to calculate the Jacobi matrix associated with a sum of measures from the Jacobi matrices of each of the measures.

We apply this method to approximate the Hessenberg matrix associated with a self-similar measure and compare it with the result obtained by a former method for self-similar measures which uses a fixed point theorem for moment matrices. Results are given for a series of classical examples of self-similar measures.

Finally, we also apply the method introduced in this paper to some examples of sums of (not self-similar) measures obtaining the exact value of the sections of the Hessenberg matrix.

Keywords: Self-similar measures; Orthogonal polynomials; Moment matrix; Hessenberg matrix

1. Introduction

In a recent work [8] we have obtained a method to approximate the moment matrix of a self-similar measure using a fixed point theorem for moment matrices. The Cholesky factorization of this moment matrix allows us to obtain an approximation of the Hessenberg matrix of the measure.

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In this paper we introduce a new method to calculate exactly the Hessenberg matrix of a sum of measures from the Hessenberg matrices of the component measures. This method extends the spectral techniques used by Mantica [17] to calculate the Jacobi matrix associated with a sum of measures from the Jacobi matrices of each of the measures (see also [5,11]).

Moreover, for the particular case of a self-similar measure μ, by iteratively applying the above method to a suitable system of measures approximating μ, we obtain a method to approximate the Hessenberg matrix associated with μ.

The study of the Hessenberg matrix associated with a self-similar measure might help to understand the structure of this measure. In [8,14], it was shown how geometric transformations of an iterated function system can be translated to transformations of moment matrices. Our method leads to similar transformations for the associated Hessenberg matrices. Our work is also related to the problem of Bernoulli convolutions [6,13,19].

In the first section of the paper we recall the concepts of self-similar measure and iterated function system (IFS) and some results about moment matrices and Hessenberg matrices that we will need in the paper.

The new methods to calculate Hessenberg matrices introduced in this paper will be presented in Sections 2 and 3. In Section 4 we will illustrate our methods with some numerical experiments.

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1.1. Moments and Hessenberg matrices

Let $\mu(z)$ be a positive measure with compact support Ω in the complex plane. Let $\mathcal{P}$ be the space of polynomials. Then, there exists a unique orthonormal polynomials sequence (ONPS) $\{P_n(z)\}_{n=0}^{\infty}$ associated with the measure μ (see [3,10] or [20]). Given two polynomials $Q(z), R(z) \in \mathcal{P}$, the expression

$$\langle Q(z), R(z)\rangle_\mu = \int_{\mathrm{Supp}(\mu)} Q(z)\overline{R(z)}\mathrm{d}\mu(z)$$

defines an inner product. Recall that we can define a hermitian moment matrix $M = (c_{jk})_{j,k=0}^{\infty}$, where $c_{jk} = \int_\Omega z^j \overline{z}^k \mathrm{d}\mu,\ j,k \in \mathbb{Z}_+ . M$ is the matrix of the inner product in the canonical basis. We denote by $M_n = (c_{j,k})_{j,k=0}^{n-1}$ the nth-section of the matrix M.

In the space $\mathcal{P}^2(\mu)$, closure of the polynomials space $\mathcal{P}$, we consider the multiplication by z operator. Let $D = (d_{jk})_{j,k=0}^{\infty}$ be the infinite upper Hessenberg matrix of this operator in the basis of ONPS $\{P_n(z)\}_{n=0}^{\infty}$, hence

$$zP_n(z) = \sum_{k=0}^{n+1} d_{k,n} P_k(z), \quad n \geq 0, \tag{1}$$

with $P_0(z) = 1$ when $c_{00} = 1$.

This Hessenberg matrix D is the natural generalization of the tridiagonal Jacobi matrix to the complex plane. The matrices M and D are related by the formula $D = T^H S_R T^{-H}$, where T is the infinite matrix whose nth-section is the lower triangular matrix, with real diagonal, obtained from the Cholesky factorization of the nth-section $M_n = T_n T_n^H$ of the moment matrix M, the superscript H applied to a matrix denotes its conjugate transpose matrix, and S_R is the shift-right matrix which is null everywhere with the exception of a subdiagonal of ones.

For more information on this subject see the books [3,20] by Chihara and Szegö, respectively.