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Conditions for the absolute convergence of Fourier integrals

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Abstract

New sufficient conditions for the representation of a function via an absolutely convergent Fourier integral are obtained in the paper. In the main result, this is controlled by the behavior near infinity of both the function and its derivative. This result is extended to any dimension $d \geq 2$.

Keywords: Fourier integral; Fourier multiplier; Vitali variation

1. Introduction

If

$$f(y) = \int_{\mathbb{R}^d} g(x) \mathrm{e}^{\mathrm{i}(x,y)} \mathrm{d}x, \quad g \in L_1(\mathbb{R}^d),$$

we write $f \in A(\mathbb{R}^d)$, with $\|f\|_A = \|g\|_{L_1(\mathbb{R}^d)}$. The possibility of representing a function via an absolutely convergent Fourier integral has been studied by many mathematicians, and is of importance in various problems of analysis. For example, the belonging of a function $m(x)$ to $A(\mathbb{R}^d)$ makes it an $L_1 \to L_1$ Fourier multiplier (or, equivalently, $L_\infty \to L_\infty$ Fourier multiplier),

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written $m \in M_1$ ($m \in M_\infty$, respectively). One such m attracted much attention in the 1950–70s (see, e.g., [5] and [15, Ch. 4, 7.4], and references therein):

$$m(x) = \theta(x)\frac{\mathrm{e}^{\mathrm{i}|x|^\alpha}}{|x|^\beta}, \tag{1.1}$$

where θ is a C^∞ function on $\mathbb{R}^d$, which vanishes near zero, and equals 1 outside a bounded set, and $\alpha, \beta > 0$. It is known that, for $d \geq 2$,

(I) if $\frac{\beta}{\alpha} > \frac{d}{2}$, then $m \in M_1(M_\infty)$;
(II) if $\frac{\beta}{\alpha} < \frac{d}{2}$, then $m \notin M_1(M_\infty)$.

The first assertion holds true for $d = 1$ as well, while the second one holds only when $\alpha \neq 1$; however, the case $\alpha = d = 1$ is obvious.

New sufficient conditions of such type in $\mathbb{R}^d$, $d \geq 1$, are obtained in this paper. Their strength and sharpness will be checked on (I) and (II).

Prior to formulating our main results in Section 2, we explain how the paper is organized and fix certain notation and conventions. In Section 3, known results on the representability of a function as an absolutely convergent Fourier integral are given as a supplement to those surveyed in [14]. The proofs of the new results are given in Section 4.

We shall denote absolute constants by c or maybe by c with various subscripts, such as c_1, c_2, etc., while $\gamma(\ldots)$ will denote positive quantities depending only on the arguments indicated in the parentheses. We shall also use the notation $\int_{\to 0}$ to indicate that the integral is understood as improper in a neighborhood of the origin, that is, as $\lim_{\delta\to 0+}\int_\delta$.

2. Main results

We start with the case $d = 1$.

Let $f \in C_0(\mathbb{R})$, that is, $f \in C(\mathbb{R})$ and $\lim f(t) = 0$ as $|t| \to \infty$, and let f be locally absolutely continuous on $\mathbb{R} \setminus \{0\}$.

Theorem 2.1. *Let* $f_0(t) = \sup_{|s|\geq|t|} |f(s)|$.

(a) *Let* f' *be essentially bounded out of any neighborhood of zero and* $f_1(t) = \operatorname{ess\,sup}_{|s|\geq|t|>0} |f'(s)|$. *If, in addition,*

$$A_1 = \int_0^1 f_1(t)\ln\frac{2}{t}\,\mathrm{d}t < \infty \quad \text{and} \quad A_{01} = \int_1^\infty \left(\int_t^\infty f_0(s)f_1(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\frac{\mathrm{d}t}{t} < \infty,$$

then $f \in A(\mathbb{R})$, *with* $\|f\|_A \leq c(A_1 + A_{01})$.

(b) *Let* f' *be not bounded near infinity,* $f_\infty(t) = \operatorname{ess\,sup}_{0<|s|\leq|t|}|f'(s)|$ *and* $f(t) = 0$ *when* $|t| \leq 2\pi$, *with* $f_\infty(4\pi) > 0$. *If, in addition, there exists* $\delta \in (0,1)$ *such that*

$$A_\delta^{1+\delta} = \sup_{t\geq 2\pi} t\, f_0^\delta(t)\, f_\infty(t+2\pi) < \infty,$$

then $f \in A(\mathbb{R})$ *and* $\|f\|_A \leq \gamma(\delta)A_\delta(1 + A_\delta^{\frac{1}{\delta}}(f_\infty(4\pi))^{-\frac{1}{\delta}})$.

The conditions of this theorem differ from known sufficient conditions in the way that, near infinity, the combined behavior of both the function and its derivative comes into play (see also the corollary below). Conditions for f_0 and f_1 in (a) are not necessary in general, but they are for