Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Journal of Approximation Theory 163 (2011) 460–480

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

A new approach to the asymptotics of Sobolev type orthogonal polynomials

M. Alfaro[a], J.J. Moreno–Balcázar[b], A. Peña[a,*], M.L. Rezola[a]

[a] *Departamento de Matemáticas and IUMA, Universidad de Zaragoza, Spain*
[b] *Departamento de Estadística y Matemática Aplicada and ICI, Universidad de Almería, Spain*

Received 10 September 2010; received in revised form 10 November 2010; accepted 15 November 2010
Available online 7 December 2010

Communicated by Guillermo López Lagomasino

Abstract

This paper deals with Mehler–Heine type asymptotic formulas for the so-called discrete Sobolev orthogonal polynomials whose continuous part is given by Laguerre and generalized Hermite measures. We use a new approach which allows to solve the problem when the discrete part contains an arbitrary (finite) number of mass points.

Keywords: Laguerre and Hermite polynomials; Asymptotics; Mehler–Heine type formulas; Zeros; Bessel functions

1. Introduction

Let $\{\mu_i\}_{i=0}^r$ be finite positive Borel measures supported on the real line. Define the Sobolev space

$$W^{2,r}(\mu_0, \mu_1, \ldots, \mu_r) := \left\{ f : \int |f|^2 \mathrm{d}\mu_0 + \sum_{i=1}^{r} \int |f^{(i)}|^2 \mathrm{d}\mu_i < +\infty \right\}$$

with the inner product

$$(f, g) = \int fg \mathrm{d}\mu_0 + \sum_{i=1}^{r} \int f^{(i)} g^{(i)} \mathrm{d}\mu_i .$$

It is very well known that this inner product is nonstandard; that is, $(xf, g) \neq (f, xg)$. Consequently, some nice properties of standard orthogonal polynomials (for example, the three-term

* Corresponding address: Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain.
E-mail address: anap@unizar.es (A. Peña).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.11.005

recurrence relation and the interlacing properties of zeros) are lost. More importantly, some powerful methods and techniques developed through the years to treat standard orthogonal polynomials have not found their equivalent in this setting and many questions remain unanswered.

Here, we give a general solution to one of those problems. Let μ be a finite positive Borel measure supported on the real line, $c \in \mathbb{R}$ and $M_i \geq 0$ for $i = 0, 1, \ldots, r$. We consider an inner product of the form

$$(f, g) = \int f(x)g(x)\mathrm{d}\mu(x) + \sum_{i=0}^{r} M_i f^{(i)}(c)g^{(i)}(c),$$

and let $\{Q_n\}_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials. Such products are called discrete types. More general discrete type products, in which derivatives of different orders are multiplied, have also been studied. Recently in [13] the authors prove that every symmetric bilinear form of this type can be reduced to the diagonal case; therefore, to some extent, we are considering the most general situation (as long as the reduction is plausible).

Our aim is to obtain Mehler–Heine asymptotic formulas for the sequence of Sobolev orthogonal polynomials when the measure which appears in the continuous part is Laguerre or generalized Hermite. We do this by comparing the Sobolev orthogonal polynomial and its classical counterpart, and see how the addition of derivatives in the inner product affects the orthogonal system. Some applications of Sobolev discrete type orthogonality within the theory of standard orthogonal polynomials are known. For instance, some standard classical polynomials with nonstandard parameters are not orthogonal in the usual sense, but they are orthogonal with respect to nonhermitian inner products (see e.g. [11] for the Laguerre case and [10] for the Jacobi case). Moreover, they are also orthogonal with respect to a discrete type Sobolev inner product (see e.g. [1] or [12]). This last approach has its origin in a paper by Gonchar where he studies the convergence of diagonal Padé approximation to meromorphic Markov type functions (see [7]). Its first use in the context of discrete type Sobolev orthogonal polynomials (and more general) dates to [14].

This idea allows to reinterpret Sobolev orthogonality as standard quasi-orthogonality (where some orthogonality conditions are lost). Consequently, the polynomial Q_n can be expressed as a linear combination (with a fixed number of terms) of standard orthogonal polynomials R_n corresponding to the modified measure $\mathrm{d}\nu = (x - c)^{r+1}\mathrm{d}\mu$; that is,

$$Q_n(x) = \sum_{j=0}^{r+1} a_n^j R_{n-j}(x). \tag{1}$$

This approach has proved to be fruitful when μ has compact support and c lies in the complement of the support of the measure.

In the bounded case, a straightforward argument allows to prove that all the connection coefficients a_n^j are bounded. If the measure μ is in the Nevai class, the orthogonal polynomials R_n have ratio asymptotic which simplifies the study of (1), in order to get the relative asymptotics of Q_n/R_n. The situation is quite different in the case of measures with unbounded support. For example, consider the Laguerre probability measure, i.e. $\mathrm{d}\mu(x) = \frac{x^\alpha \mathrm{e}^{-x}}{\Gamma(\alpha+1)}\mathrm{d}x$ with $\alpha > -1$, and the inner product

$$(f, g)_r = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty f(x)g(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0), \tag{2}$$