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Growth properties of Nevanlinna matrices for rational moment problems

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Abstract

We consider rational moment problems on the real line with their associated orthogonal rational functions. There exists a Nevanlinna-type parameterization relating to the problem, with associated Nevanlinna matrices of functions having singularities in the closure of the set of poles of the rational functions belonging to the problem. We prove results related to the growth at the singularities of the functions in a Nevanlinna matrix, and in particular provide bounds on the growth analogous to the corresponding result in the classical polynomial case, when the number of singularities is finite.

Keywords: Rational moment problems; Orthogonal rational functions; Nevanlinna parameterization; Riesz-type criterion; Growth estimates

1. Introduction

We use the following notation. $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}}$ the one-point compactification of $\mathbb{C}$ (the extended complex plane), $\mathbb{R}$ the real line, $\hat{\mathbb{R}}$ the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$, $\mathbb{U}$ the open upper half-plane, and $\hat{\mathbb{U}}$ the closure of $\mathbb{U}$ in $\hat{\mathbb{C}}$.

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A function f is called a *Pick function* if it is holomorphic in $\mathbb{U}$ and maps $\mathbb{U}$ into $\hat{\mathbb{U}}$. A Pick function is either a constant in $\hat{\mathbb{R}}$ or maps $\mathbb{U}$ into $\mathbb{U}$.

Let μ be a finite positive measure on $\mathbb{R}$. The *Stieltjes transform* S_μ of μ is defined as

$$S_\mu(z) = \int_{\mathbb{R}} C(t,z)\mathrm{d}\mu(t), \quad C(t,z) = \frac{1}{t-z}.$$

The *Herglotz–Riesz–Nevanlinna transform* Ω_μ of μ is defined as

$$\Omega_\mu(z) = \int_{\mathbb{R}} D(t,z)\mathrm{d}\mu(t), \quad D(t,z) = \frac{1+tz}{t-z}.$$

Both of these functions are Pick functions. Furthermore,

$$\Omega_\mu(z) = (1+z^2)S_\mu(z) + z\int_{\mathbb{R}} \mathrm{d}\mu(t).$$

Thus for fixed z there is a one-to-one correspondence between Ω_μ and S_μ as functions of μ.

Let M be a Hermitian, positive definite linear functional on the space $\mathcal{P}$ of polynomials, and define its moments c_n by $c_n = M[z^n], n = 0, 1, 2, \ldots$. A solution of the *Hamburger moment problem* for $\{c_n\}$ (or M) is a positive measure μ on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}} t^n \mathrm{d}\mu(t) = c_n$ for all n. (Such measures exist.) A moment problem is called *determinate* if it has exactly one solution, and *indeterminate* if it has more than one solutions.

There is a one-to-one correspondence between all Pick functions f and all solutions μ of an indeterminate problem given by

$$S_\mu(z) = -\frac{A(z)f(z) - C(z)}{B(z)f(z) - D(z)}.$$

(*Nevanlinna parameterization* of the solutions). Here A, B, C, D are entire transcendent functions where the growth is restricted as follows. Let F be any of the functions A, B, C, D. Then, for every positive ε, there exists a constant $M(\varepsilon)$ such that

$$|F(z)| \le M(\varepsilon)\exp\{\varepsilon|z|\}.$$

(Thus the function is of at most minimal type of order 1.)

For detailed treatments of important aspects of the Hamburger moment problem, see, e.g. [1,3–5,11–13,17,23–27].

The *strong Hamburger moment problem* is analogous to the classical problem, with the space of polynomials replaced by the space of Laurent polynomials (linear combinations of $z^k, k = 0, \pm1, \pm2, \ldots$). A similar parameterization of the set of solutions of an indeterminate problem holds, with the appropriate functions A, B, C, D holomorphic in $\mathbb{C} \setminus \{0\}$. When F is any of the functions A, B, C, D, there exist, for every positive ε, constants $M_\infty(\varepsilon)$ and $M_0(\varepsilon)$ such that

$$|F(z)| \le M_\infty(\varepsilon)\exp(\varepsilon|z|) \quad \text{and} \quad |F(z)| \le M_0(\varepsilon)\exp(\varepsilon/|z|).$$

For detailed treatments on the theory of strong Hamburger moment problems, see, e.g., [14,18–22].

In this paper, we treat a *rational moment problem*, where polynomials are replaced by rational functions with prescribed poles in $\hat{\mathbb{R}}$. A Nevanlinna parameterization for solutions of an indeterminate problem in terms of Ω_μ and Pick functions was proved by Almendral in [2, Theorem 9].

The classical Hamburger moment problem is a special case of the rational problem under consideration (since polynomials are rational functions with all their poles at infinity). Thus