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# Orthogonality of Jacobi and Laguerre polynomials for general parameters via the Hadamard finite part

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**Abstract**

Orthogonality of the Jacobi and Laguerre polynomials, $P_n^{(\alpha,\beta)}$ and $L_n^{(\alpha)}$, is established for $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_-$, $\alpha + \beta \neq -2, -3, \ldots$ using the Hadamard finite part of the integral which gives their orthogonality in the classical cases. Riemann–Hilbert problems that these polynomials satisfy are found.

The results are formally similar to the ones in the classical case (when $\Re\alpha, \Re\beta > -1$).


*Keywords:* Jacobi polynomials; Laguerre polynomials; Orthogonality for general parameters; Riemann–Hilbert problems; Hadamard finite part

## 1. Introduction

Orthogonality of classical polynomials is key for the study of many properties of these polynomials and their applications, and it is important to find constructive formulas for the bilinear functional that gives orthogonality.

Such formulas have been usually obtained by taking the analytic continuation (in the parameters) of the Borel measure of the classical case. Carlson used an integral kernel to establish this continuation and he proves the existence of Jacobi series for general parameters [1]. More recently Kuijlaars, Martínez-Finkelshtein and Orive find by analytic continuation that orthogonality of Jacobi polynomials can be established in some cases by integration on special

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paths in the complex plane; they also derive an associated Riemann–Hilbert problem [9]; in other cases incomplete or quasi-orthogonality is found, or even multiple orthogonality conditions (see also [10]).

In the present paper analytic continuation is established using the Hadamard finite part of (possibly) divergent integrals. Since these can be manipulated much like integrals, the classical formulas which are analytic in the parameters are formally similar. Orthogonality of the Laguerre polynomials $L_n^{(\alpha)}$ is obtained in Section 3.1 for $\alpha \in \mathbb{C} \setminus \mathbb{Z}_-$. Orthogonality of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ is established in Section 4.1 for $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_-$, $\alpha + \beta \neq -2, -3, \ldots$.

The existence of an associated Riemann–Hilbert problem for polynomials orthogonal with respect to a Borel measure on the line, introduced in [5], is now a well known result and technique which has proved very useful in deducing properties of these polynomials - see [4,7,8]. Associated Riemann–Hilbert problems in the present generalized context are found for the Laguerre polynomials in Section 3.2 and for the Jacobi polynomials in Section 4.2.

It should be mentioned that once analyticity in parameters of the Hadamard finite part is established (in Section 2.2) the remaining results follow by analytic continuation (direct proofs are given, just as a confirmation, in the Appendix). However, classical boundary conditions of the associated Riemann–Hilbert problems are not formulated in analytic terms for non-positive parameters (see [7,8]) and need reformulation to ensure uniqueness of the solution.

## 2. The Hadamard finite part of integrals $\int_0^x t^{\alpha-1} f(t)\mathrm{d}t$

The concept of the *finite part* of a (possibly divergent) integral was introduced by Hadamard [6] as a convenient way to express solutions of differential equations. He showed that this finite part of an integral (which coincides with the usual value if the integral is convergent) can be combined and manipulated in much the same way as usual integrals: they are additive on the interval of integration, changes of variable are allowed, etc. (They do not behave well with respect to inequalities.) The finite part can be calculated either by using Taylor series, or by integration along closed paths in the complex plane.

Subsequently the Hadamard finite part has been interpreted in terms of distributions (see, e.g., [11]) and it turned out that many problems of mathematical physics have solutions expressible as the Hadamard finite part of (divergent) integrals, and numerical methods of calculations have been subsequently developed (see for example [3]).

The present section contains some properties of the Hadamard finite part of integrals of the type $\int_0^x t^{\alpha-1} f(t)\,\mathrm{d}t$ with $f$ analytic at 0; when $\Re\alpha \leq 0$, $\alpha \notin (-\mathbb{N})$, its Hadamard finite part is denoted here by

$$\int_0^x t^{\alpha-1} f(t)\,\mathrm{d}t.$$

### 2.1. Notation

$\mathbb{N} = \{0, 1, 2, \ldots\}$. $\mathbb{Z}_- = \{-1, -2, -3, \ldots\}$.

### 2.2. Analyticity in $\alpha$

For $r > 0$ denote by $D_0$ the disk

$$D_0 = \{x \in \mathbb{C}; |x| < r\}.$$