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Optimal adaptive computations in the Jaffard algebra and localized frames

Stephan Dahlke[a], Massimo Fornasier[b,*], Karlheinz Gröchenig[c]

[a] *FB 12 Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein Strasse Lahnberge, D-35032 Marburg, Germany*
[b] *Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria*
[c] *Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090, Wien, Austria*

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Abstract

We study the numerical solution of infinite matrix equations $\mathbf{Au} = \mathbf{f}$ for a matrix $\mathbf{A}$ in the Jaffard algebra. These matrices appear naturally via frame discretizations in many applications such as Gabor analysis, sampling theory, and quasi-diagonalization of pseudo-differential operators in the weighted Sjöstrand class. The proposed algorithm has two main features: firstly, it converges to the solution with quasi-optimal order and complexity with respect to classes of localized vectors; secondly, in addition to ℓ^2-convergence, the algorithm converges automatically in some stronger norms of weighted ℓ^p-spaces. As an application we approximate the canonical dual frame of a localized frame and show that this approximation is again a frame, and even an atomic decomposition for a class of associated Banach spaces. The main tools are taken from adaptive algorithms, from the theory of localized frames, and the special Banach algebra properties of the Jaffard algebra.

Keywords: Adaptive scheme; Jaffard algebra; Frames in Banach spaces; Best approximation; Localization of frames; Sparse matrix

* Corresponding author.
E-mail addresses: dahlke@mathematik.uni-marburg.de (S. Dahlke), massimo.fornasier@oeaw.ac.at (M. Fornasier), karlheinz.groechenig@univie.ac.at (K. Gröchenig).

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1. Introduction

Fast matrix computations use either structure or sparsity. Structure, as used for the FFT or Toeplitz solvers, is more rigid and works only in very specific applications. Sparsity is more flexible and arises often in the discretization of operator equations with respect to a suitable basis. Roughly speaking, a matrix is sparse if each row and each column contain only few non-zero entries (or few large entries). Likewise, a vector is sparse if it has only few non-zero (or large) coefficients. The resulting matrix–vector multiplication is cheap because the operation count is determined by the number of large entries of the matrix and the vector. This observation is the key to the recent development of adaptive algorithms for the solution of infinite matrix equations by Cohen, Dahmen, and DeVore [9,10].

The numerical analysis of Cohen, Dahmen, and DeVore [9,10] was carried out with the motivation of discretizing elliptic operator equations with wavelet bases. For specific operators such as elliptic partial differential operators, the matrix with respect to a wavelet basis, the so-called stiffness matrix, is sparse. These authors then gave a rigorous analysis of the complexity of adaptive algorithms in the presence of sparsity. One of the main results guarantees that the adaptive algorithm of [9] converges with the optimal order and optimal numerical complexity.

Following the theoretical analysis of [9,10] the adaptive numerical methods have been implemented and applied successfully for the solution of operator equations coming from elliptic PDE or integral equations [3,12]. A further step was the use of (wavelet) frames instead of bases in adaptive algorithms [13–15,27]. Frames provide stable and redundant (non-orthogonal) expansions in a Hilbert space. The introduction of frames in the algorithms of Cohen, Dahmen, and DeVore was motivated by the flexible and relatively easy construction of wavelet frames on domains or manifolds, quite in contrast to the difficulty of the corresponding basis constructions. However, the use of frames led to a new problem: the resulting stiffness matrix may be singular, and at first glance one has to solve a singular equation. This problem was settled in [14,27], where it was shown that the adaptive strategies developed in [9,10] can be generalized to the frame case and maintain their advantages. In particular in [13,15] two of us contributed to showing that these adaptive algorithms based on frame discretizations are robust and perform optimally in practice.

Like [9,10], this paper is devoted to the theoretical analysis of adaptive strategies for the solution of sparse operator equations with a strong emphasis on frames. We are also motivated by applications, but they are taken not from PDE, but from time–frequency analysis and wireless communication. The emphasis in this paper is on approximation theory, and for reasons of length we have to postpone issues of implementation and simulation. Nevertheless we want to stress that the expected numerical performances will be by no means significantly different from those shown in our previous contributions [13,15] in the wavelet setting, especially in terms of robustness.

The first innovation is the chosen measure of sparsity. Whereas the adaptive wavelet schemes work with matrices in the Lemarié algebra [14,26], we investigate the analogous situation for the Jaffard algebra [25]. The sparsity of a matrix in the Jaffard algebra is given by the rate of its (polynomial) off-diagonal decay. This setting arises quite naturally in many applications, notably in time–frequency analysis [19], sampling theory [1], and in the discretization and almost-diagonalization of pseudo-differential operators in the weighted Sjöstrand class [20]. Such operators play a fundamental role in modelling wireless transmission channels in mobile communication [28].

Adaptive algorithms contain certain principal subroutines, such as the optimal truncation of an infinite vector to a finite one or the approximation of an infinite matrix–vector multiplication. We