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Slow convergence of sequences of linear operators I: Almost arbitrarily slow convergence

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Abstract

We study the *rate of convergence* of a sequence of linear operators that converges *pointwise* to a linear operator. Our main interest is in characterizing the slowest type of pointwise convergence possible. A sequence of linear operators (L_n) is said to converge to a linear operator L *arbitrarily slowly* (resp., *almost arbitrarily slowly*) provided that (L_n) converges to L pointwise, and for each sequence of real numbers $(\phi(n))$ converging to 0, there exists a point $x = x_\phi$ such that $\|L_n(x) - L(x)\| \geq \phi(n)$ for all n (resp., for infinitely many n). The main result in this paper is a "lethargy" theorem that characterizes almost arbitrarily slow convergence. It states (Theorem 3.1) that a sequence of linear operators converges almost arbitrarily slowly if and only if it converges pointwise, but not in norm. The Lethargy Theorem is then applied to show that a large class of polynomial operators (e.g., Bernstein, Hermite–Fejer, Landau, Fejer, and Jackson operators) all converge almost arbitrarily slowly to the identity operator. It is also shown that all the classical quadrature rules (e.g., the composite Trapezoidal Rule, composite Simpson's Rule, and Gaussian quadrature) converge almost arbitrarily slowly to the integration functional.

In the second part of this paper, Deutsch and Hundal (2010) [5], we make a similar study of arbitrarily slow convergence.

Keywords: Arbitrarily slow convergence; Bernstein polynomial operators; Hermite–Fejer operators; Landau operators; Fejer operators; Jackson operators; The Trapezoidal Rule; Simpson's Rule; Gaussian quadrature

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1. Introduction

There are some important algorithms in analysis that are all special cases of the following type. Let (L_n) be sequence of bounded linear operators from one normed linear space X to another Y, and suppose that the sequence converges *pointwise* to a bounded linear operator L, that is,

$$L(x) := \lim_{n\to\infty} L_n(x) \quad \text{for each } x \in X.$$

A natural and practical question that arises then is: What can be said about the *rate* of this convergence? This is an interesting and important question that does not seem to have been studied in a systematic way before. *It is the object of this paper and its sequel* [5] *to make such a theoretical study, along with numerous practical applications, with the main emphasis on convergence that is "arbitrarily slow".*

In Section 2 we give precise definitions of what it means for the sequence (L_n) to converge to L "linearly" or "(almost) arbitrarily slowly" along with other types of convergence, and exhibit some relationships between these various types of convergence. The phrase "arbitrarily slow convergence" has appeared in several papers. But in many of these, no precise definition was given. But even the precise definitions differed in a significant way. However, Schock [10] did give such a definition for a special class of methods for obtaining approximate solutions to a particular linear operator equation. In Section 2 we extend his definition to our more general setting and show that his definition is equivalent to what we have called "almost arbitrarily slow" convergence (Lemma 2.11). (The sequence (L_n) is said to converge to L *almost arbitrarily slowly* if and only if (L_n) converges to L pointwise and, for each sequence of real numbers $(\phi(n))$ with $\phi(n) \to 0$, there exists $x = x_\phi \in X$ such that $\|L_n(x) - L(x)\| \geq \phi(n)$ for infinitely many n.)

The main result of Section 3 is a "lethargy" theorem (Theorem 3.1). It *characterizes* those sequences that converge almost arbitrarily slowly. Briefly, the characterization states that the sequence converges almost arbitrarily slowly if and only if it converges pointwise, but not in norm. In Section 4, Theorem 3.1 is applied to show that the Bernstein, Hermite–Fejer, Landau, Fejer, and Jackson operators all converge almost arbitrarily slowly to the identity operator. In fact, the Bernstein and Hermite–Fejer operators even converge arbitrarily slowly to the identity operator. In Section 5, Theorem 3.1 is used to show that all the classical numerical quadrature rules (e.g., the composite Trapezoidal Rule, the composite Simpson's Rule, and Gaussian quadrature) all converge almost arbitrarily slowly to the definite integral functional.

The notation and terminology is standard and can be found, e.g., in [3].

2. Types of convergence

In this section, we assume that $X(\neq \{0\})$ and Y are normed linear spaces over the same scalar field and let $\mathcal{B}(X, Y)$ denote the normed linear space of all bounded linear operators L from X to Y with the usual norm (the operator norm)

$$\|L\| := \sup_{x\neq 0} \frac{\|L(x)\|}{\|x\|},$$

where the same notation is used for the norm in X, Y, and $\mathcal{B}(X, Y)$. Let the sequence (L_n) and L be in $\mathcal{B}(X, Y)$.

First it is convenient to recall various types of convergence.